WHAT YOUR COLLEAGUES ARE SAYING . . .

"Who better than Pam Harris to help you introduce K–2 students to mathematical reasoning—the language, the music, and the poetry of mathematics. A must-read book filled with teaching strategies and creative ideas."

Jo Boaler
Nomellini & Olivier Professor of Education, Stanford University
Stanford, CA

"The abilities to count and to add are foundational to mathematics. All that follows is built upon these cornerstones. Get it right and math becomes 'figure-out-able.' In this book, Harris gives us the tools to get it right. Through real classroom examples, Harris takes us through strategies that are easy to adopt and effective in getting and keeping students engaged in the work of understanding mathematics."

Peter Liljedahl
Professor, Simon Fraser University;
Author, *Building Thinking Classrooms*
Vancouver, British Columbia, Canada

"It is with great enthusiasm that I endorse this transformative book. At the heart of this work is a compelling discussion of reasoning. Through rich narratives and classroom vignettes, we see that math fact fluency is not only figure-out-able but enjoyable—sparking curiosity and confidence in every student. In short, this book is a masterclass in making math fact fluency meaningful and accessible for all."

Dr. Nicki Newton
Math Consultant, Newton Education Solutions
Bridgeport, CT

"This book is a gift to primary teachers. It offers clear ideas we can use right away to help students build real understanding and develop as mathematical thinkers. From counting to additive reasoning, and through the power of models and Problem Strings, this book supports teachers in making instruction more purposeful and responsive."

Graham Fletcher
Math Specialist
Atlanta, GA

"Finally, the book that K–2 educators have been waiting for is here! Harris wrote a book that explores the complexity of foundational numeracy skills and shares research-based approaches to develop mathematical reasoning with our youngest learners. This book will not only help teachers cultivate curiosity and confidence and build a community of mathers, but it will also help teachers become the mathers they were always meant to be."

Deborah Peart Crayton
CEO & Queen Mather, My Mathematical Mind
Author, *Readers Read. Writers Write. Mathers Math!*
Charlotte, NC

"This book empowers K–2 teachers to deeply explore and understand the mathematics of the early grades, equipping them to foster students' mathematical reasoning and conceptual understanding. Grounded in research and filled with practical classroom examples, it offers valuable guidance for transforming early elementary math instruction."

Marria Carrington
Director of Mathematics Leadership Programs, Mount Holyoke College
South Hadley, MA

"What a powerful resource for equipping mathematics educators with the knowledge and skills necessary to develop their students' mathematical reasoning and promote their confidence as mathematicians!"

Janet D. Nuzzie
District Intervention Specialist, K–12 Mathematics,
Pasadena Independent School District
Pasadena, TX

"If you've ever heard Pam Harris talk about the Levels of Sophistication in Mathematical Reasoning and thought, *'I get it but what does that look like with MY students??'* This book delivers! It reveals how young children develop counting skills and thinking strategies for mathematical operations. You'll gain insights into student development, practical tasks to showcase their thinking, and modeling techniques that benefit every child in your classroom."

Christina Tondevold
The Recovering Traditionalist
Founder, Build Math Minds
Orofino, ID

"Pam Harris offers teachers another invaluable resource for transitioning from repetitive procedural instruction to helping students develop deeper, more conceptual understanding of mathematics. Through numerous examples of student thinking about mathematical concepts in the K–2 classroom, she provides extensive treatment of the ways children approach problem solving. This comprehensive approach will certainly prove helpful for teachers who are developing their understanding of how students learn to make sense of mathematics independently."

John Tapper
CEO, All Learners Network Inc.
Burlington, VT

"This is a must-read for all K–2 teachers. Harris helps teachers understand how they can set their young learners up for long-term success in the math classroom. This book provides teachers with a resource that combines content knowledge, high quality instructional practices and ready-made instructional materials all in one! Upper elementary, middle school and even high school math teachers would all benefit from reading this book and better understanding the development of mathematical reasoning, too."

Brandon Pelter
Mathematics Teacher, Bridgeport Public Schools
Norwalk, CT

"This book is a game-changer for educators. With clear examples and practical strategies, it empowers teachers to transform their instructional practices. Packed with 'aha' moments and insights into the teaching and learning of mathematics, it inspires confidence in all of us. This is a must-read for anyone looking to make math instruction more meaningful for students while supporting the teacher with the 'why' behind it all."

Jennifer Lempp
Author and Educational Consultant
Alexandria, VA

"Math reasoning kicks algorithms to the curb when students engage their brains (not just their pencils) to solve problem strings using a variety of strategies. Harris takes the guess work out of teaching computation strategies intentionally by providing problem strings, teaching tips, and sentence frames for beginning and experienced teachers."

Carrie S. Cutler
Clinical Associate Professor of Mathematics Education,
University of Houston
The Woodlands, TX

"This latest book from Pam Harris takes research related to the major milestones of mathematical development in the primary grades and transforms it into a language that is easy to understand. Using carefully chosen examples, real world experiences, and student voices, Harris has written a book that is illuminating and practical. Primary educators, whether new to the profession or seasoned experts, will find ideas here that resonate and challenge them to listen closely in order to further thinking."

David Woodward

Founder and President, Forefront Education

Lafayette, CO

Developing MATHEMATICAL REASONING

The Strategies, Models, and Lessons to Teach the Big Ideas in Grades K–2

PAMELA WEBER HARRIS

Contributing Writers: **CAMERON HARRIS, KIM MONTAGUE, KOURTNEY LAMBERT PETERS**

CORWIN Mathematics

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Contents

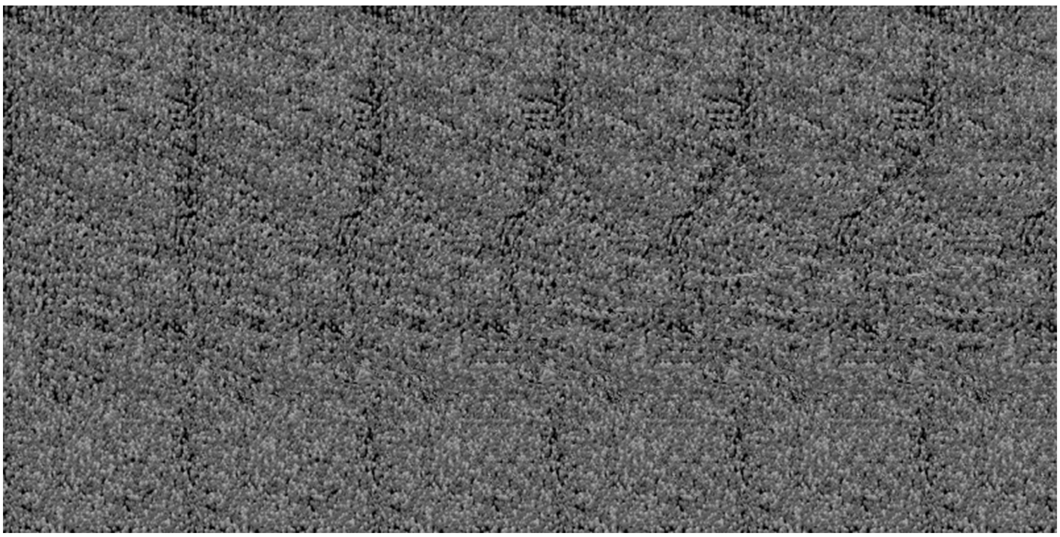
Note From the Publisher: The authors have provided video and web content throughout the book that is available to you through QR (quick response) codes. To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

Videos may also be accessed at **mathisfigureoutable.com/dmrcompanionK-2**

Preface

Trying to learn real math in a fake math classroom is a lot like looking at an autostereogram.

Autostereograms are pictures made of colored dots that at first glance look like visual noise. If you have the skill to focus your eyes just right, that visual noise resolves into a three-dimensional image that will appear to float in front of the page.



Source: Adapted from https://en.wikipedia.org/wiki/File:Stereogram_Tut_Random_Dot_Shark.png with CC Attribution-Share Alike 3.0. Retrieved May 17, 2025.

In this example, focusing beyond the page about three inches will reveal the image of a shark. Some of you will see the shark immediately, but many of you will not. Some will struggle and eventually see the shark. But some of you will probably never see the shark.

Think of this autostereogram of a shark as learning math.

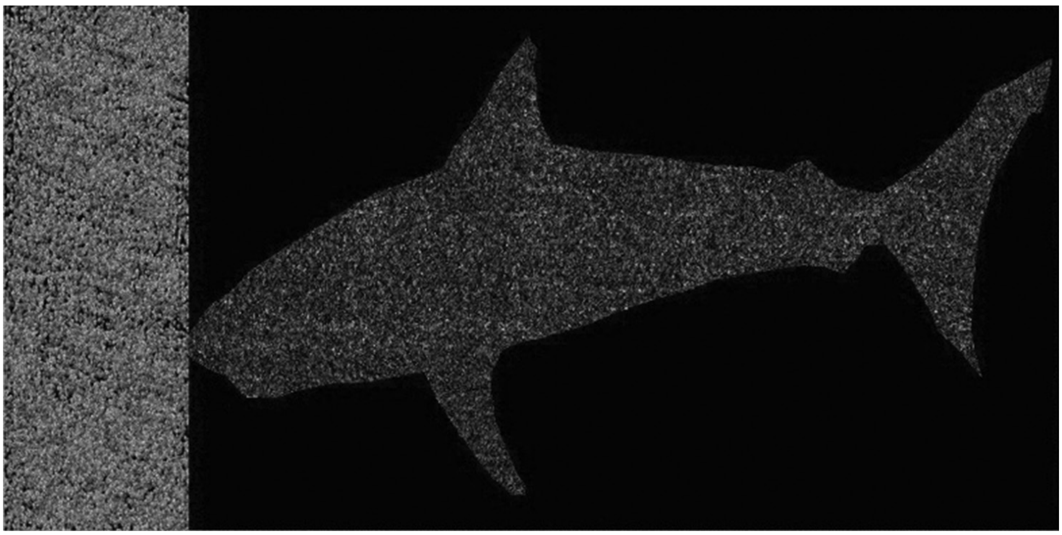
Some of you will say, "I have no idea what's going on. The teacher is calling this a shark, but it doesn't look like what I thought a shark is. They say to memorize these weird squiggles, and bits of dots, so that must be what a shark is—I guess I will memorize these five squiggles and this horizontal wave—that must be what a shark is. I'll memorize that so that when I see it later, I'll know it's a shark." This was me.

Others, without guidance from a teacher, have the natural inclination to very quickly focus their eyes to see the shark. "Right there is a shark; surely everyone can see the shark. It's obviously right there. I wonder why the teacher is talking about five squiggles and the horizontal wave. Weird. It's just a shark." This was my son.

Still others can't see the shark but aren't willing or able to play the game the first group does. The nonsensical memorization of lines and dots when there is supposed to be a *shark* isn't enough for them to hold on to. Math becomes increasingly stressful. Students become convinced they just "aren't math people."

Who can blame them? Not me, not when we tell them there is a shark in the water, and then when they say they can't see it, we just throw more and more confetti dots at them.

The following is part of the 3-D image seen when viewed correctly.



Source: Adapted from https://en.wikipedia.org/wiki/File:Stereogram_Tut_Random_Dot_Shark.png with CC Attribution-Share Alike 3.0.

A lot of really good work in the mathematics education field today focuses on getting students to be willing to try to focus on the shark again. *Building Thinking Classrooms* by Peter Liljedahl (2021) falls into this category. After years of nonsense, many students aren't willing to try to make sense of anything. Disruptions to the classroom norm like vertical-nonpermanent-surfaces and randomly chosen groups (as Liljedahl recommends) can do wonders. The willingness to try squinting (focusing) again is crucial.

But that's not enough. All students need guided, deliberate instruction aimed at how to see the shark. They need focused, purposeful instruction to do the mental actions that math-ers (Crayton, 2025) do. That's where this book comes in.

When I say math is figure-out-able, I'm saying everyone can be taught to see the shark. Everyone can be taught to see real math, to math their world.

When I say math is figure-out-able, I'm saying everyone can be taught to see the shark.

Some 25 years ago, as I was beginning to understand that math was figure-out-able myself, I sat down with my friend Mary. She was a second-grade teacher who taught near Austin, Texas. I had been reading about how mathematicians naturally have many ways to solve problems and an intuition for picking the approach that will be far easier and more efficient.

So I asked, "How do you think about adding 38 and 29?"

She added them by lining up the digits by place value in vertical columns, the traditional North American addition algorithm. Adding this way required "regrouping," handling digit overflow when the added digits come out to more than 9.

$$\begin{array}{r} {}^{1} \\ 38 \\ +\,29 \\ \hline 67 \end{array}$$

I asked if she could think of any other way.

She looked at me as if I were an alien. Then, after a long pause, she lined up the numbers in columns again but with the order reversed.

$$\begin{array}{r} {}^{1} \\ 38 \\ +\,29 \\ \hline 67 \end{array} \qquad \begin{array}{r} 29 \\ +\,38 \\ \hline \end{array}$$

Not quite what I meant. I took that to indicate that she had not yet had the opportunity to think about multi-digit addition outside the confines of the traditional algorithm. To her (and to young me) addition meant doing the steps. Excited to show her what I'd been learning, I began listing off other ways.

Judging by her rising panic, this was not helping.

The strategies I demonstrated were good strategies, but my teaching approach needed lots of work. Showing the strategies didn't give Mary any experience building the relationships. The strategies seemed like more algorithms Mary thought I was telling her to memorize. More lists of squiggles to see in an autostereogram. In other words, I discovered that giving a prescriptive list of strategies is a bad way to build the mental connections people need to actually use strategies as strategies.

In my early years of teaching, I started to realize that what I *thought* was math was only a shadow of what math really *is*.

The inkling that I was missing something massive had started when my eldest son—a first grader—started exploring math in ways I never had—even as a high school mathematics teacher. This drove me to dive into every bit of existing research I could get my hands on.

I began to realize that the math I had learned and the math I taught my high school students was more akin to trivia with extra steps than real mathematical reasoning. We memorized when to use which procedure and lists of steps, and we memorized the answers to pieces of those steps. Undoubtedly, some of my students were like my son and managed to piece together the logic and patterns behind those steps and really reason mathematically, but most of us, myself included, did not.

Some of you reading this are surprised that a high school mathematics teacher was not actually math-ing. Others are not surprised but are very frustrated. You know math makes sense, and has an underlying logic, but you don't understand why your teachers refused to teach you that logic. You might be wondering if the reason your teachers didn't teach you the underlying logic is because they didn't know that logic exists. Still others are like me, who bought into the myth that to do math was to rote-memorize and mimic. Students who tried to hold on to dozens of algorithms and formulas and, without really understanding why many of them work, try to pick the right one to get right answers.

If you are in this last group, you might also be starting to get an itch that math isn't quite what you thought it was. This can be very unsettling. It gave me something of an early mid-life crisis. Maybe you aren't in any of those groups. Maybe you know that math makes sense, that it is figure-out-able—that math taught as figure-out-able engages and enlightens. What you may not know is how to teach it that way.

If as a student you rocked at memorizing squiggles, then as a teacher you probably rock at teaching memorization of squiggles with rhymes, games, and/or drills. Your students have gotten the best possible start you could have given them. But most, like yourself, have not yet seen the actual shark in the water. You want to see the shark and help your students math. And now that you'll know better, you can do better.

If as a student you saw the shark, powerful and attention demanding, then as a teacher you probably started out confused about why most of your students didn't see what you saw. Maybe you found ways to help with that, maybe you didn't. Most likely driven by good intentions and frustration, you ended up teaching the same way you had been taught and accepting that only a rare few students would understand math the way you do. You want to help students see the beautiful sharks and are hungry for effective ways.

If as a student you had the looming, insistent feeling that there was something else in the water with you, and were occasionally freaked out that your teacher refused to acknowledge a fin cutting through the surf or explain the bite marks in your surfboard, then as a teacher you know something is up. If you knew the story about math your teachers were trying to sell you didn't add up, and were frustrated by their refusal to acknowledge that, then you want to improve. So as a teacher you do your best, sharing what you have been able to piece together. And you wish you could do more.

Whatever group you may fall into, this book will help you teach more real math to more students.

Even twenty-five years ago there was much research on this subject, but finding the good stuff and figuring out how to piece it together took time, active investigation, experimentation, and a background in higher math to sort the usable from the less useful.

I've found the single most powerful way to get students willing to squint (focus) again is to provide them with an experience where they see the shark. Where their squinting (focusing) is rewarded.

Real math-ing consists of reasoning using connections, understandings, and relationships. Fake math-ing is memorizing disconnected sets of facts and mimicking procedures, where each adds yet another ball of confusion to be juggled on top of the last one.

Real math-ing consists of reasoning using connections, understandings, and relationships.

This book is the guide to getting Kindergarten, first-, and second-grade students seeing sharks. It is the result of the research and experimentation I have done in the last twenty-five years to learn how to build students' brains to do more math, rather than merely burden those brains with more disconnected sequences of steps.

ABOUT THIS BOOK

Chapter 1 summarizes the contents of the first book in this series, *Developing Mathematical Reasoning: Avoiding the Trap of Algorithms* (Harris, 2025a). While it is intended for that book to be read before this one, this chapter serves as an abbreviated launch point or refresher. In short, algorithms have numerous shortcomings when used as teaching tools.

Chapters 2 and 3 lay out the critical foundation for all mathematics found in counting and Counting Strategies. Understanding counting has relevance at all grade levels, as we are either teaching it or watching for students who have not yet grown to also use Additive Reasoning.

Chapters 4 through 7 focus on what is required to develop more and more sophisticated additive reasoning, moving through the major strategies for addition within 20, subtraction within 20, multi-digit addition, and multi-digit subtraction. These major strategies are the "what to do instead" of teaching algorithms.

Chapters 8 and 9 move on to discuss the "how" to teach of Chapters 4 through 7's "what."

Chapter 8 details the different kinds of lessons to use.

Chapter 9 is about models and modeling, crucial tools and actions necessary to drawing out the patterns in student thinking needed for purposeful instruction in mathematical reasoning.

Chapter 10 concludes with how to begin in the classroom. Where to start and with what.

Each chapter includes tips and frequently asked questions throughout as well as actions the reader can take—either personal exercises or things to try in class.

Corwin and I will be publishing three additional grade-specific companion books (3–5, 6–8, and 9–12) on a sixth-month cadence once this book is released, which will offer more ideas, more practice, and more practical advice, concentrated specifically on each grade band. These books will be complementary to the anchor volume and this book.

Neither this book nor the rest of the (currently!) planned series have a focus on geometry, data, or measurement. Due to time and space constraints, I choose to restrict this series to the fundamentals of mathematical reasoning, as they are foundational to meaningful geometry, data, and measurement.

LANGUAGE USE IN THIS BOOK

There are a few terms that will be helpful to parse out before beginning.

- **Mathematical Reasoning**

  As used in this book, the term *mathematical reasoning* does not mean just a general ability to think. This is not a fuzzy "think better" approach that doesn't include doing the math and getting results. Mathematical reasoning is about building stronger brains and expects more, not less from students, giving them the tools to be successful at math-ing. It demands increasing sophistication of strategy. We meet students where they are and then develop from there. For example, students will not only know their addition facts, they will actually own them and be able to use the relationships in problems. It includes content-specific milestones such as understanding of place value, addition and subtraction, and so forth.

- **Sophistication**

  Sophistication as used in this book is a descriptor of how different levels of mathematical reasoning relate to each other. This includes how the domains of reasoning relate to each other, such as Additive Reasoning being more sophisticated than counting but also how individual strategies within a domain relate to each other. For example, both the Get to 10 and the Give and Take major strategies use Additive Reasoning. However, the latter requires more simultaneity and developed understanding.

  Sophistication is always a descriptor of a thought process, never a descriptor of a thinker.

- **Problem Strings**

  Problem Strings are everywhere in this book, because Problem Strings are the single best teaching routine for building sensemaking and teaching the major strategies. Problem Strings are deliberately ordered sequences of problems designed to develop an important big idea, model, or strategy. They are always meant to be teacher-led routines, with the teacher facilitating learning by revealing each problem one at a time, drawing out—modeling—student thinking, and crafting class discussions after each problem to highlight the patterns at work for the given strategy.

  For a more detailed look at Problem Strings for each grade level, and more than 200 example strings each, see *Numeracy Problem Strings: Kindergarten* (Harris 2025c), *First Grade* (Harris 2025b), and *Second Grade* (Harris 2024). Problem Strings will also be discussed in greater detail in Chapter 7.

- **Strategy and Model**

  The term *strategy* can mean instructional strategy or general problem solving strategy, but in this book, strategy means how you use mathematical relationships to reason through a problem. This is different than how you represent that strategy—that is a model.

  The term *model* is commonly used with many different definitions in mathematics teaching. In this book, model means "representing student thinking," which relates to the "representation of student thinking" and also as a "tool for thinking."

  For more on the differences between *strategy* and *model*, see Chapter 9. The full breakdown is reserved for the end of the book, as the context given by the proceeding chapters will aid greatly in understanding it.

Acknowledgments

- A mighty thanks to my son, Cameron Harris, for helping me find real math-ing and write this book.
- Thank you to Kim Montague and Kourtney Lambert Peters for also helping write this book—your expertise and attention to detail are greatly appreciated. It was fun! Thanks, Kim, for the great video footage! It's been fantastic working on three major projects with the two of you all at The. Same. Time! We mostly kept it all straight, right?
- Thank you to Sue (digits or numerals—that is the question!) and Kira and the rest of the gang at Math Is Figure-Out-Able for keeping the ship sailing while my head is down writing or I'm traveling. And thank you to Ann Latham for her meticulous work with permissions.
- Thanks to my husband, kids, grandkids, parents, and siblings who make life worth living and excellence worth striving for. And for the chocolate.
- A hearty thank you to Senior Acquisitions Editor Debbie Hardin and the Corwin crew for making this whole project work with a smile. Two down, three to go!
- Many thanks to Stephanie Lugo, Sarah Hempel, and Melisa Williams for inviting me into your classrooms to film. You and your students rock!
- And to the parents who allowed us to interview your three-, four-, and five-year-olds, I appreciate you and your kids!

With great respect and gratitude, I acknowledge the thousands of primary teachers and their students who have let me in their classrooms and taken our workshops—online and in person—and learned along with me. Your dedication to your craft is admirable. I hope you find this book helpful in your continued journey. You are my favorite group of teachers to work with because if I can convince you, you'll actually do it!

And to the author of life, thank you, Lord, for giving me a message worth sharing and the people around me to help me get it out.

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About the Author

Pamela Weber Harris is changing the way we view and teach mathematics. Pam is the author of several books, including the *Numeracy Problem Strings K–5* series, *Building Powerful Numeracy,* and the *Foundations for Strategies* series. As a mom, a former high school math teacher, a university lecturer, and an author, she believes everyone can do more math when it is based in reasoning rather than rote-memorizing or mimicking. Pam has created online *Building Powerful Mathematics* workshops and presents frequently at national and international conferences. Her particular interests include teaching real math, building powerful numeracy, sequencing Rich Tasks to construct mathematics, using technology appropriately, and facilitating smart assessment and vertical connectivity in curricula in schools PK–12. Pam helps leaders and teachers make the shift that supports students to learn real math because math is figure-out-able!

PART I

Setting the Stage

CHAPTER 1

Mathematics for Teaching

In the early 2000s, I facilitated a workshop for K–2 teachers near Austin, Texas. I couldn't help but notice that something had stirred the proverbial hornet's nest as the teachers came in. Everyone was mad enough that I knew we would struggle to get any meaningful learning done without clearing the air. So I asked them what was going on.

It turns out that a colleague of theirs, as part of the requirements of her university master's program, had administered an IQ test to each of them.

TIP

Maybe it's not the best idea to use your colleagues as guinea pigs for your master's IQ test project. That may or may not go well.

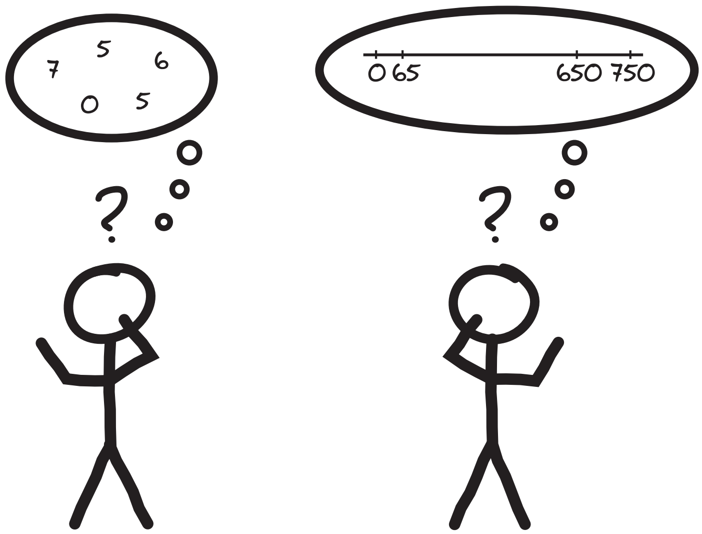
These teachers were upset because the math section of the test was "completely unfair." They thought the problems they were asked to solve, without being able to write anything down, were unreasonable.

They gave an example problem that involved "traveling 750 miles, going 65 miles per hour . . . and we don't even remember the rest because who can hold all of that in your head?!"

Something about the way they said "in your head" caught my attention, so I asked for exactly what was in their head when they heard this problem.

My question was not one they were expecting. To them, there was only one way to think about this problem so far. The digits 7, 5, 0 and then the digits 6, 5, and that it was unreasonable to expect them to hold five digits in their heads while trying to listen to the rest of the problem.

Because their sense of *number* was digits and their sense of *math*-ing was doing things with columns of digits, these teachers were not thinking about the relationship between about 750 and around 65. They weren't creating a mental map as the problem was being said, that 750 is a bit more than 10 times 65. It's not that they were refusing to consider the connections. It's as if those connections didn't exist.



This was a major epiphany for me. That, at no fault of their own, students (and teachers) could have such a limited conception of numbers.

Perhaps more importantly, working with those teachers the rest of that day I saw them develop more sophisticated mathematical reasoning. I witnessed again that anyone can *math*. Often people just don't know it's a thing. Those teachers didn't know there was an option besides lining up digits into columns and doing an algorithm.

I wished there was a book to help teachers understand the holes in their own mathematical reasoning and teach students so that those holes didn't get passed along.

It took a minute, but this is that book.

The mathematics of early grades is neither straightforward nor simplistic. As Dr. James Tanton (2019) observed, "Even the act of counting is fundamentally subtle and nuanced. There is so much intellectual richness to probe and explore there." This is true for the teaching of early grade mathematics. In Grades K–2, there are foundational concepts and benchmarks of reasoning that are not obvious to recognize or simple to teach, even if we ourselves have long since learned the content of those grades.

How can we do this teaching of real math-ing? Check out this first-grade classroom lesson.

Melisa, a first-grade teacher, starts the lesson by commenting that this Problem String is connected to their recent work where they used 10 and adjusted. Melisa asks Romie to repeat what the student had said previously.

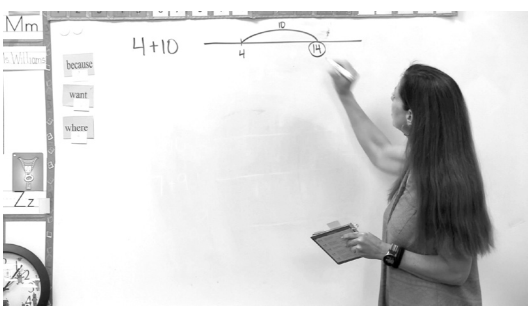
Romie responds, "Tens are helping us with our nines."

Melisa says and writes the first problem, reminding students to signal when they have an answer with a thumbs up: 4 + 10.

"Addison, what is that one?" asks Melisa.

"Um, it equals 14."

Melisa responds as she draws a number line, "We've been working on that for a long time, haven't we?"



She asks, "Let's see how that can help us with the next problem: 4 + 9. Gabi, what do you think?"

Gabi responds with long pauses, “Um I think it equals . . . um . . . ”

Melisa suggests that she look up at the first problem to see if the connection helps.

Still tentative, Gabi tries “Is it 13?”

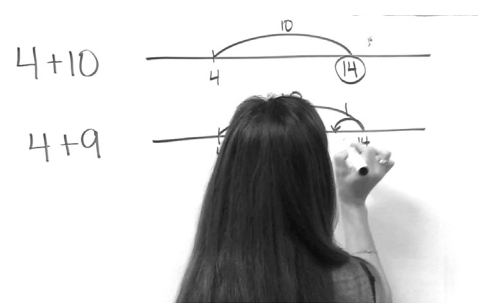
Melisa gives a neutral response, not cueing Gabi if her answer is correct, “Ok, why do you think it’s 13?”

Gabi, far more confident now, responds without pausing, “I think it’s 13 because if you take away 1 from 14, it equals 13.”

“Nicely done. That was really well explained,” says Melisa, and she represents Gabi’s thinking by drawing a new number line with a jump of 10 and then a jump back of 1. Melisa’s words and modeling focus the students’ attention on Gabi’s thinking:

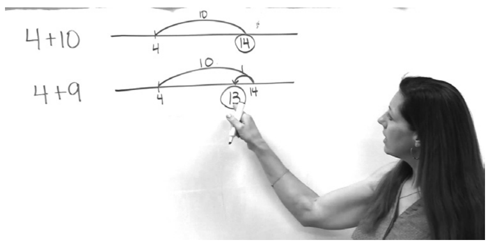
- Rather than “correct” or “right answer” Melisa says, “Nicely done,” and “well explained.”
- She makes Gabi’s thinking visible on a model that provides Gabi with a visual-spatial representation that can assist her to reflect on her own thought process.
- This number line model makes Gabi’s thinking (and the target strategy) more accessible to the class, and because her thinking is point-at-able, it’s more discuss-able.

As Melisa draws the jump back 1, she asks, “And Gabi, why did we have to jump back one?”



Gabi responds, “Because the 10 turned into a 9.”

Melisa points to the number line: "Yes, it went down 1, didn't it? Because it was 10," pointing to the first equation, "and now it's 9," pointing to the second equation.



"You explained that one very well. Good job." Melisa moves on to the next purposefully planned question, "7 + 10. Thumbs up when you know this one."

You can almost see the gears turning, and then Jayson sits up tall with his thumb proudly up. Melisa calls on him, "Jayson, what is 7 + 10?"

Jayson answers, "17."

"Very nice," Melisa says as she draws a number line to represent the problem. To help students focus on the important relationships, she repeats the problem as she draws, "7 + 10, we've been working a lot on that one, haven't we? 7 + 10 is 17," and circles the 17.

She continues, "Let's see how 7 + 10 can help us with the next one. Some of you probably already know what the next one is going to be." She writes 7 + 9 on the board.

Melisa just gave two different nudges with two different goals:

- "How can it help with the next one" can help students potentially realize that they can use + 10 to help with + 9.

- "You might know what the next one will be" can help challenge those students who might have already been using the strategy. They get a chance to predict what the next problem could be based on the pattern, giving the rest of the students space to solve the problem.

Melisa asks, "Ben?"

Ben responds, "Uh . . . 16."

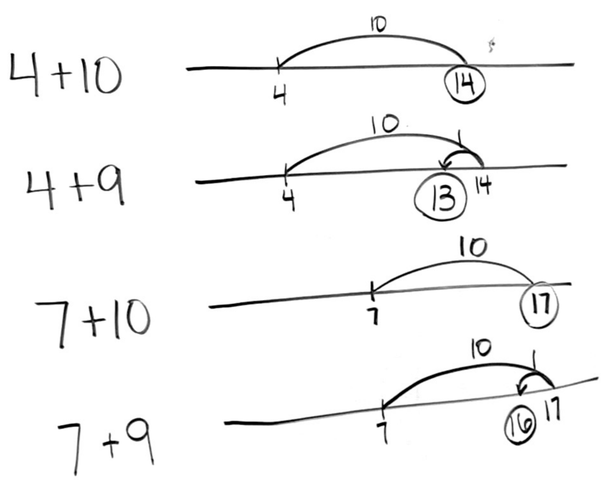
"Why?" Melisa prods.

Ben smiles, "Because, like, 7 plus 10, but 1 less."

As Melisa represents his thinking, she says, "So 7 plus 10, that's what our first problem was." She points to the number line above, "and that was 17. But you said, we had to go . . ."

Ben finishes, "Back one."

As she draws the jump back one, Melisa asks, "Why did we have to go back one again?"



Ben answers, "To make 16."

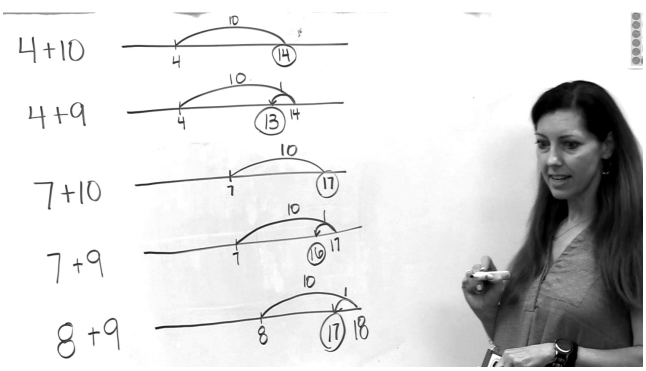
"And also, this one was 9, right?" Melisa points to the current problem, "And this one was 10, and we had to go back 1 to get our 9, right?"

When Ben responds "yes," Melisa changes the pattern by asking a problem without a helper first. "Okay, good job, let's check out this last one, last one for this string." And she writes: 8 + 9.

After seeing many thumbs up, Melisa asks, “Troy, what do you think?”

“Uh, 8 + 9 equals . . . ” Troy begins, “If it was a 10, it’d be 18, but, it’s a 9. It would just be taking 1 away. And if you take 1 away, it would be 17.”

Melisa restates his words while she draws a number line to represent his thinking, “So you said, if it was 10, that would be 18, but we have to take away that 1 because it’s actually a 9 we are adding. . . . So we landed on 17. Wow. look at that. You did so great with that!”



Watch this clip to see reasoning in Melisa’s classroom.

https://qrs.ly/3zgl252

To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

In this Problem String, we see students concentrating, smiling, justifying their thinking. As important, they are building and using mathematical connections. These students are *math*-ing.

WHAT'S THE PURPOSE OF LEARNING MATH?

For a moment, leave the world of education as it currently exists. If you had to argue for the inclusion of mathematics in K–12 education in any form, what arguments would you make?

I argue that mathematics and literacy are the two most powerful tools a person can have for understanding, organizing, and making an impact on the world. From there, a number of consequences logically flow. First, nothing, no matter how much it uses the trappings of mathematics, is worth spending valuable classroom time on if it doesn't aid in a student's ability to understand, organize, and make an impact on the world. Second, among possible ways to spend limited classroom time, priority must be given to those activities and approaches that make the greatest impact.

I argue that mathematics and literacy are the two most powerful tools a person can have for understanding, organizing, and making an impact on the world.

Which means, third, that in the best-case scenario, algorithms and step-by-step procedures as answer-getting tools are near the very bottom of the barrel of ways to spend classroom time. In the worst case, teaching to mimic the steps of algorithms can actively inhibit many students' ability to understand, organize, and make an impact.

Let's return to the opening hypothetical to illustrate why. If your argument for the inclusion of mathematics in K–12 education is that students need to be able to get the answers to math problems as fast and easily as possible, math education ceases to need to be a pillar of the K–12 experience and can instead be reduced to a few weeks covering how to use a calculator.

FREQUENTLY ASKED QUESTIONS

Q: But Pam, that's what math is. The definition of math is to rote-memorize and mimic steps.

A: If you're thinking that, it's understandable. I invite you to consider that you might have been trapped by algorithms. Take heart! This book will help you get free.

THE DEVELOPMENT OF MATHEMATICAL REASONING

Our goal in math instruction should be to help students develop *mathematical reasoning*, which includes content. This is not some fuzzy *think better* game. Instead it is helping students learn to logically reason using mathematical concepts, strategies, models, and properties. Reasoning mathematically is solving problems using what you know and, in the process, building more and more real math.

Reasoning mathematically is solving problems using what you know and, in the process, building more and more real math.

There is a hierarchy of reasoning domains defined by increasing levels of sophistication and simultaneity. I call it the *Development of Mathematical Reasoning* (Harris, 2025). This represents the high-level hierarchy of milestones students must progress through to develop reasoning-based proficiency.

To reason through a problem, a student must grapple with multiple levels of complexity simultaneously. As students develop their brains, they create schema—ways of structuring their mental maps so that they can focus on the big picture or drill down to the specifics. Because they own this relationship map, their ability to grapple with multiple things will increase, leading to more efficiency and understanding.

As students develop their brains, they create schema—ways of structuring their mental map so that they can focus on the big picture or drill down to the specifics.

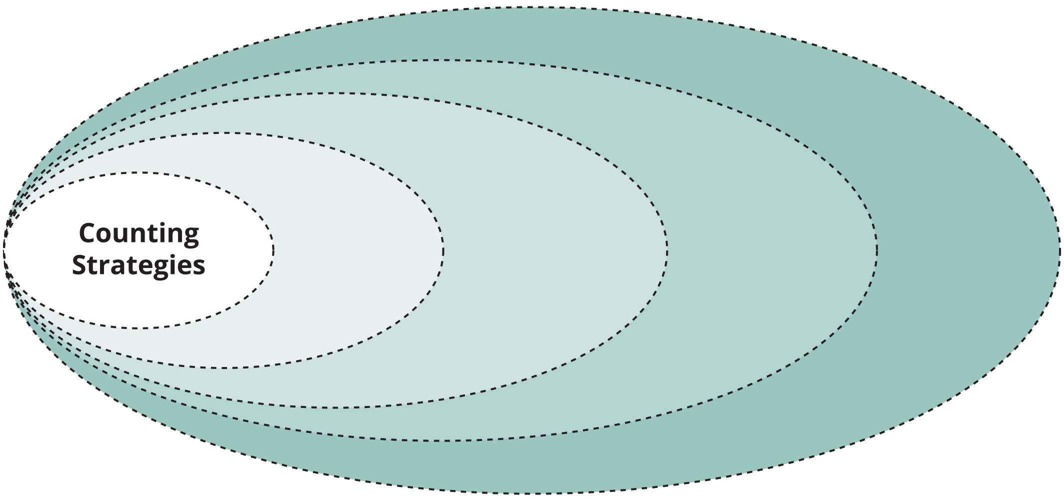
Sophistication includes the level of simultaneity employed as well as the complexity of the mathematical ideas at play.

Consider the following example of increasing simultaneity and sophistication when a student starts to Count On. Typically, for a problem like 6 + 3, the student has been counting the first set, 6, counting the second, 3, then counting them all, 9. We call this Counting Three Times. To Count On requires that the student can conceive 6-ness, to start at 6, not needing to count up to 6. This means that the student is simultaneously considering both the 6 objects in the set and that the word or numeral "6" represents the cardinality of the set (the last number in the count represents the amount). Then the student counts on from 6: 7, 8, 9. How does the student know when to stop? They must simultaneously keep track of the count and when to stop. This is cognitively difficult and takes time and experience to develop.

THE BEDROCK: COUNTING STRATEGIES

When students are beginning and solving beginner problems, they use Counting Strategies (see Figure 1.1).

FIGURE 1.1 ● The First Level of Sophistication in Mathematical Reasoning



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Using Counting Strategies entails solving problems with one-by-one counting. We can solve addition, subtraction, and even multiplication and division problems counting one by one.

TIP

Even if you are a kindergarten or first-grade teacher, it is important that you realize that the next goal is Additive Reasoning. You will almost certainly have students who are ready to develop further, so knowing those landmarks and how to build them is important.

Using Counting Strategies is more than just being able to say the counting sequence. It's about solving problems and, while solving, considering the numbers involved as sets of 1s.

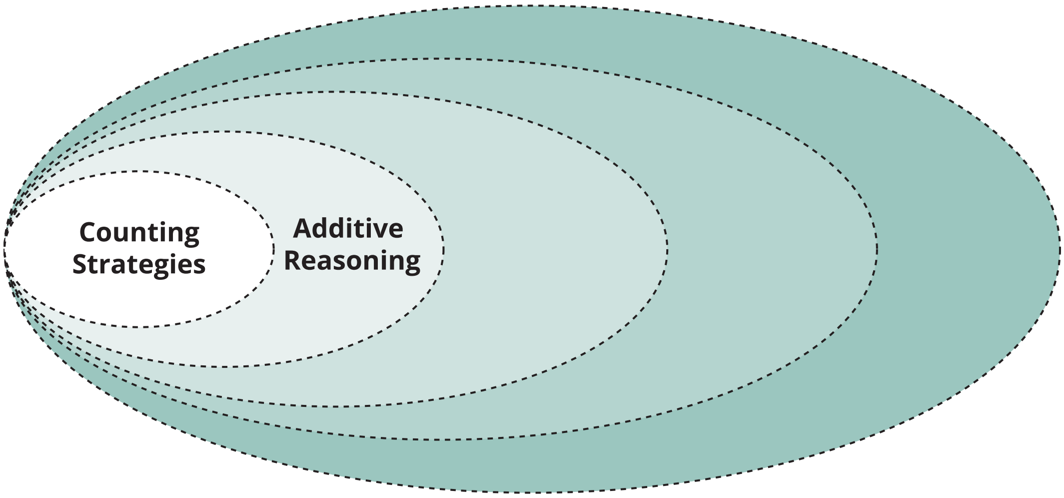
The Counting Strategies domain is the foundation of everything else. Teachers of counting are key to students developing all the domains because they begin it all. This is so important and exciting.

You will learn about major counting principles and Counting Strategies and how to develop them in Chapter 2. Develop these important Counting Strategies with an eye to then building Additive Reasoning.

THE FOUNDATION: ADDITIVE REASONING

Additive Reasoning is characterized by thinking in bigger jumps of numbers than 1 at a time (see Figure 1.2).

FIGURE 1.2 ● The Second Level of Sophistication in Mathematical Reasoning



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An additive reasoner considers numbers simultaneously as sets of 1s *and* combinations of other sets of numbers. It's about composing and decomposing numbers in additive chunks. This Additive Reasoning is the major goal of kindergarten through second grade. As you help students develop counting and Counting Strategies, you are doing so with the end goal of those students developing Additive Reasoning. This means that you are always working toward students thinking in terms of bigger jumps than 1 at a time. This takes time, effort, and many experiences.

Grades 3, 4, 5, and 6 should continue to build Additive Reasoning with bigger, smaller (decimals), and more complicated numbers.

You will learn more about this important Additive Reasoning and how to develop it in Chapters 4 through 7.

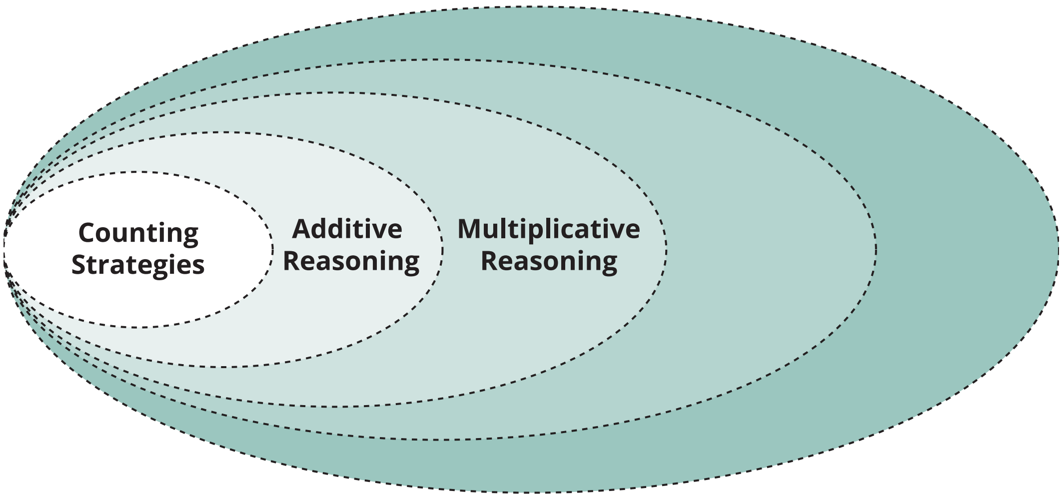
THE NEXT DOMAINS: MULTIPLICATIVE, PROPORTIONAL, FUNCTIONAL REASONING

Multiplicative Reasoning is more mentally sophisticated than Additive Reasoning.

One is reasoning multiplicatively when considering the number of groups, the number in each group, and the total all at the same time. Solving a multiplication or division problem using Additive Reasoning looks like adding or subtracting one group at a time. Solving these problems with the more sophisticated Multiplicative Reasoning means using more than one group at a time, grouping the groups.

As is shown with the ovals in Figure 1.3, Multiplicative Reasoning is built on and based on Additive Reasoning. Because of this, it is essential that K–2 teachers work to help students develop Additive Reasoning, not leaving them in Counting Strategies, even if students are getting correct answers.

FIGURE 1.3 ● The Third Level of Sophistication in Mathematical Reasoning



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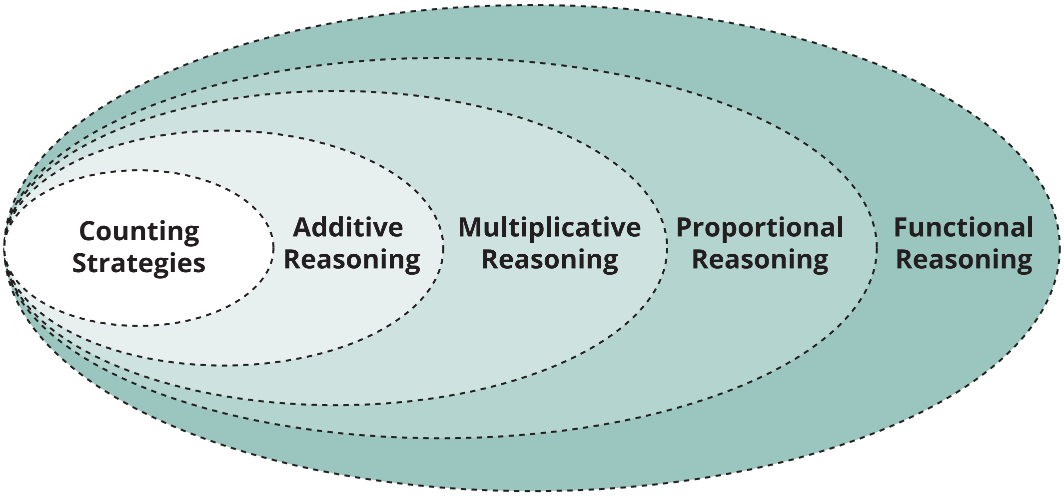
In algorithm-driven instruction, this is the last stop for most students—there is just too much to memorize and keep straight if their understanding of math is mimicking procedures. It does not have to be. With the strong Additive Reasoning you will build in your K–2 students, they will have the sophistication of thought to be able to then develop Multiplicative Reasoning.

Proportional and Functional Reasoning are the domains of middle- and high-school mathematics.

As many as 90 percent of adults in the United States are not reasoning proportionally (Lamon, 2020). The vast majority of those same adults took classes in high school algebra, geometry, and more. This means they were getting answers, but because they did not have the necessary building blocks of reasoning, they were not able to reason more sophisticatedly.

If you are interested to learn more about these domains (Figure 1.4), read my *Developing Mathematical Reasoning: Avoiding the Trap of Algorithms* (2025).

FIGURE 1.4 ● The Full Spectrum of Mathematical Reasoning



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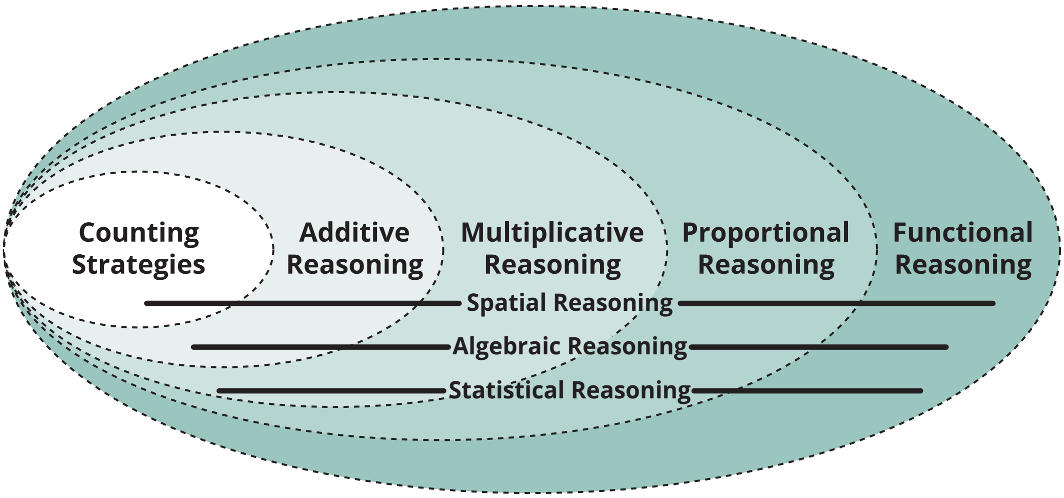
SPATIAL, ALGEBRAIC, AND STATISTICAL REASONING

In addition to the five hierarchical domains in Figure 1.4, there are three longitudinal domains: spatial, algebraic, and statistical. I refer to them as longitudinal, because unlike the five hierarchical stages, they don't build off of other domains of reasoning in a specific sequence. Each of these three should be developed in concert with the hierarchical five (see Figure 1.5).

For example, spatial reasoning should see development all the way from Counting Strategies (1:1 tagging) to Multiplicative Reasoning (area models), to Functional Reasoning (distance and area defined by functions). Spatial reasoning is all about visual, geometric relationships of shapes, dimensions, measurement, location, graphs, and trends.

Algebraic reasoning is all about generalizing and working with generalizations. In counting, this might be that whether you have 5 dogs, 5 marbles, 5 balloons, or 5 anything else, the quantity is 5. In Additive Reasoning, this could be that you can add 10 to any

FIGURE 1.5 ● The Full Spectrum of Mathematical Reasoning With Longitudinal Domains



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number and the digit in the ones place doesn't change, and the tens place goes up by 1 ten. In multiplication, that might be that 5 multiplied by anything is half of that thing multiplied by 10.

Statistical reasoning concerns data, inferences, and predictions. This is the ability to draw useful information from data sets and create meaningful conclusions. It is also the ability to recognize when someone has manipulated factual data to favor their biased conclusions. At a basic level, statistical reasoning includes representing data in pictographs (the number of people in students' families), discussing the data (many of us have four people, some have less, and some have more), and possible changes (how would the graph change if we had students move in with five people in their family?).

Any lesson or routine intended to build reasoning must be conscious of how the material intersects the five hierarchical domains of reasoning with the longitudinal three. For example, spatial number line models for students learning Counting Strategies will not be very effective if those students do not yet have the spatial reasoning to appreciate the measurement meanings behind a number line.

MAJOR STRATEGIES

These are specific approaches to problem solving that take advantage of the human mind's natural pattern-finding abilities to build a student's mathematical understanding and empower them to solve problems quickly and efficiently. We can cultivate and train mathematical intuition that allows students to engage in *math*-ing.

Unlike algorithms, the learning of strategies is synergistic. Where each new algorithm is another series of steps to potentially misremember and confuse (is it carry forward or bigger bottom better borrow?), strategies are mutually reinforcing. The more relationships you own, the more strategies you learn, which in turn builds more relationships.

This book will help you learn the important counting principles and counting strategies, the major strategies for addition and subtraction within 20, the major strategies for double-digit addition and subtraction, and how to teach them.

Note that while, for example, there are four major strategies for addition where there is generally only one addition algorithm traditionally taught, it does not take four times as long to teach four strategies as it does to teach one algorithm. Each subsequent step of an algorithm breeds confusion or presents another opportunity for error, whereas every major strategy builds off of the one before it, bringing *increased* clarity and proficiency. Every subsequent major strategy learned makes mistakes with a previous one *less* likely.

Every subsequent major strategy learned makes mistakes with a previous one less likely.

Finally, because strategies create relationships (through context-driven understanding), student retention of learning is far higher than that of algorithms—which tend to create disconnected, easily lost islands of procedure (Jensen & McConchie, 2020).

FREQUENTLY ASKED QUESTIONS

Q: Are you advocating for direct instruction or inquiry?

A: I'm advocating for a shift in goals—from mimicking algorithms to developing mathematical reasoning (which includes content). With that new goal in mind, how to teach becomes clearer: good guided inquiry for everything that is logical knowledge and clearly telling for the bit that is social/conventional knowledge. Teachers have clear mathematical goals; help students grapple *long enough*;

(Continued)

(Continued)

guide students to important generalizations through purposefully crafted discussions; anchor learning; and keep building on that learning to move the mathematics forward using tasks open enough that all students continue to have access and continue to be challenged. By doing this, students are solving problems not just correctly and efficiently but also more sophisticatedly. Students will be more successful longer. There will be more on how to do this in the rest of the book.

Conclusion

The purpose of math class is to develop mathematical reasoning, not mathematical answer-getting. What we need are not mere calculators but thinkers and do-ers of mathematics. Our role as teachers is not to have students rotely mimic algorithms that only provide answers to problems but to guide and support students as they develop as math-ers (Crayton, 2026). We can help students realize that they can use what they know to solve problems. Math is figure-out-able!

Discussion Questions

1. What was your experience as a student in math class? Did you more often reason through problems using what you know? Did you rote-memorize and mimic your teacher? How do you think that impacts the way you teach?
2. What's the difference between logical mathematical knowledge and social knowledge? Why does the difference matter?
3. What's the difference between an algorithm, a strategy, and a model? The book will further differentiate between these, but for now, how do these show up in your teaching?