Algebra

Algebra

Conceptual Category Overview

The algebra conceptual category synthesizes content from previous grade levels to deepen understanding of the structure of mathematics. At the high school level, the algebra standards extend these concepts along three lines. First, students simplify expressions (closely aligned with material from Number System). Whether the standard addresses combining like terms to create a more efficient way of writing an expression or factoring an expression to find underlying structures (such as the zeros of the associated equation), students are doing or undoing operations. That is, students are "describing a computation in general, abstracting from" arithmetic and specific instances "they understand" (CCSS-M, 2010, p. 62).

Second, students are solving equations and inequalities. This requires an understanding of work with expressions, as well as understanding what the solution set of equations or inequalities means. Finally, students look for patterns in mathematical contexts and are creating rules to describe those patterns. This makes connections between the structural aspects of algebra and functions with modeling. The algebra and function conceptual categories are closely linked. CCSS-M specifies the difference between them as that of mathematical concepts of expressions and equations on the one hand (algebra) and functions on the other (CCSS-M, 2010, p. 3).

Direct Connections to Algebra in the Middle Grades

Students have studied numbers and operations from the very first grade levels. In the middle school grades, the standards of Number and Quantity enabled students to gain proficiency in working with numerical expressions, including use of order of operations, and to explore the entire system of numbers and properties that create the real numbers. Additionally, in Grades 6 through 8, the Expressions and Equations standards allowed students to

explore linear expressions and equations as discrete entities. The Ratio and Proportion standards showed a growth from ratio, proportion, and direct variation and underpin later algebraic work. Students also began their study of functions in Grade 8, which included evaluating functions for a given y-value (that is, evaluating an expression) and identifying equivalent expressions for linear functions (simplifying or "unsimplifying" expressions).

SUGGESTED MATERIALS

A.SSE	A.APR	A.CED	A.REI	
✓	✓			Algebra tiles and other physical models to understand the terms in expressions and how to represent terms with variables and constants, create equivalent expressions, factor quadratic expressions, and complete a square and algebra tiles to understand operations on polynomials.
✓		✓	✓	A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.
		✓	✓	Dynamic geometry environments; interactive applets; handheld computation, data collection, and analysis devices; and computer-based applications are used to support students in exploring and identifying mathematical concepts and relationships.

ALGEBRA—OVERARCHING KEY VOCABULARY

A.SSE	A.APR	A.CED	A.REI	
✓	✓	✓		**Coefficient** – A numerical or constant quantity generally placed before the variable(s) in an algebraic expression. For example, 5 in 5d is a coefficient as is $\frac{2}{3}$ in $\frac{2}{3}x^2 y$.
✓	✓	✓		**Distributive property of addition over multiplication** – For any values of a, b, and c, $a(b + c) = ab + ac$.
✓	✓	✓		**Equivalent expressions** – Have the same value regardless of what is substituted for the variables in the expressions.
✓	✓	✓	✓	**Equation** – A mathematical statement that two quantities are equal, that is, they are the same as one another.
✓	✓	✓		**Exponent** – Initially, students learn that an exponent is a shorthand way to show how many times a number, called the base, is multiplied times itself; $6 \cdot 6 \cdot 6 \cdot 6$ is rewritten as 6^4. When exponents become rational numbers (other than integers) and irrational numbers, the definition expands. Eventually, exponents may be complex numbers, as well, e.g., $e^{-\pi i} + 1 = 0$.
✓	✓	✓		**Exponential expression** – An equation in which a constant base is raised to a variable exponent, such as 6^x or e^{rt}, where e is the irrational number 2.718 . . .
✓	✓			**Expression** – A record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function.
✓	✓			**Factor** – A portion of a multiplication expression, usually an integer or monomial that, when multiplied by other factors, gives the entire quantity.
✓				**Geometric series** – The summation of a geometric sequence for which the ratio of each two consecutive terms $\frac{a_{k+1}}{a_k}$ is a constant.
	✓			**Identity** – An equation that is true for all values of the variables.
		✓		**Inequality** – A mathematical statement that one expression is greater than (or equal to) or less than (or equal to) another.
			✓	**Linear equation** – An algebraic equation of the form in which there are two first degree variables that covary with each other. A typical format is $y = mx + b$ in which m and b may be any real numbers.

A.SSE	A.APR	A.CED	A.REI	
✓				**Maximum** – The greatest value in a data set or the greatest possible value of an expression.
✓				**Minimum** – The least value in a data set or the least possible value of an expression.
	✓			**Polynomial** – A mathematical expression containing only the operations of addition, subtraction, multiplication, and nonnegative integer exponents.
				Quadratic equation – A quadratic equation is written in a single variable and has exponents of 2, 1, and 0 only on the variables, such as $3x^2 - x + 1 = 5$.
✓	✓	✓		**Quadratic expression** – A polynomial expression whose greatest exponent is 2. These expressions may be written in standard form, $ax^2 + bx + c$, where $a \neq 0$.
	✓			**Rational expression** – Quotient of two polynomials with a nonzero denominator.
	✓			**Remainder Theorem** – For a polynomial $p(x)$ and a number a, the remainder when dividing $p(x)$ by $x - a$ is $p(a)$. That is, $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
		✓	✓	**System of equations** – A set of two or more equations that has a common set of solutions. Solving the system means find those common solutions.
✓				**Term** – An algebraic expression that represents only multiplication and division of variables and constants
✓	✓			**Zero of a function** – A value of the independent variable where the value of a function equals 0.

Seeing Structure in Expressions (A.SSE)

Domain Overview

An expression is a "record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function" (CCSS-M, 2010, p. 62). The standards in A.SSE focus on seeing expressions as sums of terms and products of factors and seeing complicated expressions as built up out of simpler ones.

Students work with generalizing the conventions of order of operations with numbers, such as the use of parentheses, so that identifying equivalent expressions and simplification of expressions has no ambiguity about the process used or the steps followed. One way to consider work with expressions is that it is a generalization of the rules of computation students have learned in previous grades.

Students read and understand expressions, including grasping the underlying structure of expressions. This means students find different but equivalent ways of writing expressions (for example, factoring or using exponential properties) that highlight some different facets of their meaning. For example, $p - 0.05p$ can be interpreted as a loss of 5% of the value of p. Rewriting $p - 0.05p$ as $0.95p$ shows that subtracting the loss is the same as multiplying the original value by the constant factor, 0.95.

A.SSE—KEY VOCABULARY

A.SSE.A	A.SSE.B	
✓	✓	**Coefficient** – A numerical or constant quantity generally placed before the variable(s) in an algebraic expression. For example, 5 in 5d is a coefficient as is $\frac{2}{3}$ in $\frac{2}{3}x^2y$.
✓	✓	**Distributive property of addition over multiplication** - For any values of a, b, and c, $a(b + c) = ab + ac$.
✓	✓	**Equivalent expressions** – Have the same value regardless of what is substituted for the variables in the expressions
✓	✓	**Equation** – A mathematical statement that two quantities are equal, that is, they are the same as one another.
✓	✓	**Exponent** – Initially, students learn that an exponent is a shorthand way to show how many times a number, called the base, is multiplied times itself; $6 \cdot 6 \cdot 6 \cdot 6$ is rewritten as 6^4. When exponents become rational numbers (other than integers) and irrational numbers, the definition expands. Eventually, exponents may be complex numbers, as well, e.g., $e^{-\pi i} + 1 = 0$.
✓	✓	**Expression** – A record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function.
✓	✓	**Factor** – A portion of a multiplication expression, usually an integer or monomial that, when multiplied by other factors, gives the entire quantity.
	✓	**Geometric series** – A series for which the ratio of each two consecutive terms, $\frac{a_{k+1}}{a_k}$, is a constant function of the summation index k.
	✓	**Maximum** – The greatest value in a data set or the greatest possible value of an expression.
	✓	**Minimum** – The least value in a data set or the least possible value of an expression.
	✓	**Quadratic expression** – A polynomial whose greatest exponent is 2.
	✓	**Term** – An algebraic expression that represents only multiplication and division between variables and constants.
	✓	**Zero of a function** – A value of the independent variable where the value of a function equals 0.

Notes

Algebra | Seeing Structure in Expressions
A.SSE.A

Interpret the structure of expressions.

STANDARD 1 **A.SSE.A.1**: Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.

STANDARD 2 **A.SSE.A.2:** Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Cluster A: Interpret the structure of expressions.

The middle grades standards in Expression and Equations serve as connectors from arithmetic expressions in elementary school to more complex work with algebraic expressions in high school. As the complexity of expressions increases, students recognize expressions as being built out of basic operations; that is, initially they see expressions as sums of terms and products of factors. Students also see that complicated expressions are built up out of simpler ones. That is, students recognize different terms in an expression (and understand what a term is) and can recognize one or more parts of the expression as discrete components. For example, in the expression $(x + 4)^2 - 6$, students first see a difference of a constant and a square and then see that inside the square term is the expression $x + 4$. The first way of looking at the expression tells students that the value of the expression is always greater than or equal to -6, since a square is always greater than or equal to 0. The second way tells them that the square term is zero when $x = -4$. Putting these together, students can determine that this expression attains its minimum value, -6, when $x = -4$. Students explore ways to rewrite expressions, such as combining like terms, that is, terms with exactly the same variable quantities in them, such as $2xy$ and $3xy$, but not $2x$ and $2x^2$. This process relates to the arithmetic of radical expressions, for example, $2\sqrt{xy} + 3\sqrt{xy}$ and the structure of computation. *The Progressions for the Common Core State Standards in Mathematics, High School, Algebra* (2013) stresses that simplification is not a goal of the Algebra standards, as different forms of an expression are useful in different contexts. For example, viewing the expression for a parabola in vertex form may be more useful in a situation, such as finding the extreme value, whereas the factored form is more useful for finding the zeros of the expression.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students use different forms of algebraic expressions to solve problems. They also use quantitative reasoning to understand relationships among different expressions. Even though all of the Standards for Mathematical Practice are evident in these standards, Practices 7 and 8 are particularly prominent here. Students are expected to see how the structure of an algebraic expression reveals properties of the function it defines. They are expected to move from repeated reasoning, such as using exponents to rewrite repeated multiplication with variables and numbers.

Related Content Standards
A.APR.A.1   A.APR.C.4   A.APR.D.6   N.CN.C.8

STANDARD 1 (A.SSE.A.1)

Interpret expressions that represent a quantity in terms of its context.*

  a. *Interpret parts of an expression, such as terms, factors, and coefficients.*

  b. *Interpret complicated expressions by viewing one or more of their parts as a single entity.*
     For example, interpret P$(1 + r)^n$ *as the product of* P *and a factor not depending on* P.

Students see that complicated expressions are built up out of simpler ones. Part of this understanding means students know what a factor is, know how factors and coefficients are related, and know how constants, factors, and/or coefficients relate to terms in an expression. Students work with the structure of a complicated expression, identifying the parts to help understand what the expression means. Complex formulas in science or other disciplines are a rich source of applications for these standards. The Doppler Effect formula, $f = \left(\dfrac{c + v_r}{c + v_s} \right) f_0$, can be seen as a product but also as including a part that is a quotient in which both the numerator and denominator have a similar format in which the velocity of the waves in a medium, c, are affected by the velocity of the receiver or source to the medium. Students can determine the relative values of each and make estimates about the size of the Doppler Effect. Students may also recognize that the computational rules they have learned mean that the expression $\left(\dfrac{c + v_r}{c + v_s} \right) f_0$ is not equivalent to $\left(\dfrac{1 + v_r}{1 + v_s} \right) f_0$.

What the TEACHER does:

- Requires students to understand and use vocabulary such as *factor*, *coefficient*, *term*, and like terms.
- Provides representations of expressions so students may compare terms such as $w + w$ and $2w$ in contexts—that is, $P = 2(l + w)$ and $P = 2l + 2w$. By using pictures, manipulatives, and symbols, students can make sense of these equivalent expressions.
- Uses problems such as compound interest. $P\left(1 + \dfrac{r}{12}\right)^{12n}$ so students may identify important components in context, such as $1 + \dfrac{r}{12}$ and $12n$.

What the STUDENTS do:

- Explain, in their own words, what *factor*, *coefficient*, *term*, and like terms mean in the context of expressions.
- Identify factors, coefficients, different terms, and like terms in expressions.
- Identify individual parts of an expression as a single entity to make use of the structure of the expression.

Addressing Student Misconceptions and Common Errors

Students frequently confuse the parts of an expression. A common error is assuming a constant is not a term or that an expression such as $3w + z - 4xy$ has four terms because the student is counting variables and not terms. Manipulatives and area models are one strategy to use for students with this misconception. When students have difficulty with vocabulary, having them restate a word or phrase in their own words with an example and a non-example is helpful. For example, $3x$ and $4x$ are like terms, but $3x$ and $4x^3$ are not because like terms have exactly the same variable factors regardless of the coefficient. Contextual examples of expressions, such as the Doppler Effect formula, give students an opportunity to consider what the underlying structure of an expression can tell them, helping students parse complicated expressions into their components.

Connections to Modeling

The aforementioned Doppler Effect formula is one example of a formula that may be applied in the modeling sequence.

Related Content Standards

A.SSE.A.2   A.APR.A.1   A.APR.C.4   A.APR.D.6   N.RN.A.1

Notes

STANDARD 2 (A.SSE.A.2)

Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

Students view expressions from a dynamic perspective—that is, there is progress or change when using expressions flexibly in different formats to find the form that is most useful in a contextual situation. In the standard, students consider $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$ and as the factored form $(x^2 - y^2)(x^2 + y^2)$. This can help students consider zeros and graphing patterns. Recognizing different forms of an expression and being able to apply the forms is an important step mentioned in the *Progressions for the Common Core State Standards in Mathematics for Algebra* (2013). An example in that document (p. 5) concerns looking at the expression for the sum of a series of perfect squares, $\dfrac{n(2n + 1)(n + 1)}{6}$, whose structure allows students to see the expression has a degree of 3 with a leading coefficient of $\dfrac{1}{3}$, that is the term $\dfrac{1}{3}n^3$, which is helpful when studying rules for summation notation and integration, providing an underpinning of calculus. Rewriting $y = x^2 + 2x + 1$ as a trinomial square pattern $y = (x + 1)^2$ can help students consider the graph of the function as a translation of $y = x^2$ instead of having to calculate individual ordered pairs to determine the graph.

What the TEACHER does:

- Provides problems that allow students to discover special patterns that occur with the structure of expressions, such as forms of the difference of squares for Algebra or connections of the sine and cosine when using the Pythagorean Theorem to discover $sin^2\theta + cos^2\theta = 1$.
- Uses problems such as compound interest.

$$P\left(1 + \frac{r}{12}\right)\cdot\left(1 + \frac{r}{12}\right)\cdot\left(1 + \frac{r}{12}\right)...\left(1 + \frac{r}{12}\right)(12n \; times) \; as$$

$$P\left(1 + \frac{r}{12}\right)^{12n} \; as \; P\left[\left(1 + \frac{r}{12}\right)^{12}\right]^{n} \; where \; students \; can \; see$$

the structure of $1 + \dfrac{r}{12}$ and how exponents help rewrite the expression.

What the STUDENTS do:

- Explain, in their own words, how specific structures are seen in different expressions.
- Rewrite expressions using structure to identify important components of the expression (where zeros may occur) or to end behavior.

Related Content Standards

A.SSE.A.1   A.APR.C.5   A.APR.D.7   N.RN.B.3   N.CN.C.8   F.IF.B.7.c

Addressing Student Misconceptions and Common Errors

Students need to recognize special patterns, such as difference of squares and greatest common factors, so they can use them in new situations. This means students need to have a conceptual basis for patterns, such as an area model for difference of squares. Students may struggle with finding the greatest common factor or seeing a pattern in an expression such as $4x^4 + 8x^2 + 4$. One strategy uses substitution. In the expression, students may use $2x^2 = u$ and rewrite the expression as $u^2 + 4u + 4$ and look for a pattern they know to rewrite the higher level polynomial. Substitution is a useful strategy that occurs when solving equations or systems of equations and in other contexts in mathematics.

Notes

Algebra | Seeing Structure in Expressions
A.SSE.B

Write expressions in equivalent forms to solve problems.

STANDARD 3

A.SSE.B.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

a. Factor a quadratic expression to reveal the zeros of the function it defines.

b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

c. Use the properties of exponents to transform expressions for exponential functions. For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.

STANDARD 4

A.SSE.B.4: Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments.★

Cluster B: Write expressions in equivalent forms to solve problems.

Students choose among different equivalent forms of an expression to analyze the quantity represented by the expression. Students learn several ways to solve a quadratic expression (and its related equation), whether by using quadratic formula, graphing, completing the square, or factoring. Students need to recognize that each of these methods has its appropriate place. For example, solving an easily factorable expression is an efficient way to find the zeros of a quadratic function that can also help students make connections between the factored form and the graph by making the zeros evident in the symbolic form. Completing the square leads to the form $y = a(x - h)^2 + k$, which helps students determine whether the quadratic opens up or down and which helps students identify the vertex (and whether it is a maximum or minimum). Completing the square also may be used to derive the quadratic formula so that students understand how the formula connects to the expression. Students need an understanding of these forms and when to use them. A student does not need to complete the square to find the vertex of $y = x^2 + 2x + 1$, nor is multiplying $x(x + 1)(x - 2) = 0$ and then factoring an efficient method. The derivation of the sum of a finite geometric series can be a direct consequence of using equivalent forms of expressions, where $(x - 1)(x^{n-1} + x^{n-2} + \ldots + x + 1) = x^n - 1$ may be used.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 2. Use quantitative reasoning.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students construct viable arguments as to whether expressions are equivalent or not. Relating equivalent expressions also connects to the use of mathematical structure. Deriving the formula for a finite geometric series may involve looking for and expressing repeated reasoning. The use of equivalent expressions and formulas may be a part of modeling and quantitative reasoning. Tasks that ask students to generalize patterns, use geometric series, and explain properties of expressions all relate to precision, communication, and problem solving.

Related Content Standards
F.IF.C.8   F.LE.A.2   G.GPE.A.1   G.GPE.A.2   G.GPE.A.3

Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

a. *Factor a quadratic expression to reveal the zeros of the function it defines.*

b. *Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.*

c. *Use the properties of exponents to transform expressions for exponential functions. For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Students learn several ways to analyze quadratic expressions and their related functions. This standard focuses on factoring and completing the square. Each of these methods has its appropriate place. Students understand that factoring may be an efficient way to analyze a quadratic expression. Completing the square yields another form of a quadratic expression that is sometimes called the vertex or graphing form. This form, $y = a(x - h)^2 + k$, makes it relatively easy to find the vertex of the quadratic function and apply its form to transformations from geometry. Students also apply the properties of exponents to create equivalent expressions that can give insight into the quantity described by the exponential expression.

What the TEACHER does:

- Provides problems that require different forms of quadratic expressions for their solution (factoring to find roots and determining the extreme value by completing the square).
- Requires students to explain their solution processes and why the form of an expression they used was appropriate.
- Provides contextual examples that require equivalent forms of an expression for a solution or for analysis (such as making an expression to represent the volume of a box formed when cutting corners from a piece of paper, then folding the paper to make a box, and then finding the maximum volume).
- Uses exponential decay growth applications to highlight cases in which exponential properties may be used.
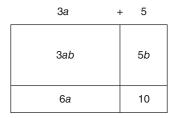
What the STUDENTS do:

- Explain how they used equivalent forms of quadratic expressions to determine important components of a quadratic function (and its graph).
- Solve contextual problems using equivalent forms of expressions. For example, students may find extrema, end behavior, growth factors, or decay factors.

Addressing Student Misconceptions and Common Errors

Students may initially have difficulty learning methods for factoring. It is beneficial to connect factoring with functions. Students may start by looking at the graph of $y = x - 3$ and determining the zero, slope, and y-intercept. Next students look at a graph with both $f(x) = x - 3$ and $g(x) = x + 1$. After determining the zeros, students determine what the product of the graph is by using the y-values of sets of ordered pairs. For example, $f(0) = -3$ and $g(0) = 1$, so the product of $f(0)$ and $g(0)$ is -3. After finding several sets of products, students plot the points and sketch the graph of the function. Then, students compare the graph and the equivalent version of the product, $y = x^2 - 2x - 3$. Students may explore one or two other examples to see the connection of the graphs and the quadratic function. The next step is to have students consider the function $h(x) = x^2 + 4x - 5$ given that $h(x) = f(x)g(x)$ and $f(x) = x - 1$. By using ordered pairs, graphing, or whatever solution process they find useful, students can determine the missing function, $g(x)$, is $x + 5$. This has set up a connection among graphs, zeros, and factors for the students.

Another method that may be helpful for students when factoring is to first consider multiplication of binomials using an area model. Algebra tiles or drawings are both helpful.

The area model can be reversed so that students are asked to find the missing factors for a given polynomial. Students, then, make connections among the quadratic symbolic form and the factored form, as well as with the graphing situations mentioned earlier. In the figure below, students are to find the missing factor for $3ab + 6a + 2b + 10$ given one factor is $3a + 5$.

$3a$	$+$	5
$3ab$		$5b$
$6a$		10

The use of multiple representations when approaching factoring, as well as viewing factoring as undoing multiplication, helps students grasp how the use of the distributive property and like terms in multiplying is reversed with factoring. It is important to make a connection to graphing zeros so students have a concrete reason for learning to factor rather than viewing factoring as a discrete skill with no purpose other than itself.

A similar use of multiple representations aids students learning to complete the square. Students learn the process of completing the square to find zeros but can then later apply the method to finding the center of a circle (or ellipse) and to finding key features of conic sections in the Geometry standards. It is important for students to consider completing the square pictorially as well as symbolically. (See F.IF.C.8 for further discussion.)

Though students have worked with rules of exponents in Grade 8, the sophisticated approach of transforming functions is an extension of their knowledge. That is, students are now going beyond rewriting repeated multiplication with exponents but are looking at ways to write equivalent expressions that extend their use of the laws of exponents they explored. Contextual examples help students comprehend the use of rules, the most accessible being half-life and compound interest.

Connections to Modeling

Students may use completing the square when considering elliptical modeling with an echo chamber or to find the focus (position on the filament) in a car headlight. Finding the zeros for a model that created a projectile motion equation may also require students to rewrite a quadratic in an equivalent form.

Related Content Standards
N.RN.B.3   F.IF.C.7c   F.IF.C.8   G.GPE.A.1   G.GPE.A.2   G.GPE.A.3

Notes

STANDARD 4 (A.SSE.B.4)

Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments.*

Students consider geometric sequences in their study of functions (F.LE.A.2: Construct . . . geometric sequences, given a graph, a description of a relationship). Finding the sum of a finite geometric series connects to some contextual problems (the mortgage payment context) and extends student thinking about how sequences and series are related and used.

One way to derive the formula for the sum of a geometric series is to use this identity:

$$(x - 1)(x^{n-1} + x^{n-2} + \ldots + x + 1) = x^n - 1.$$

A geometric series is written as $a + ar + ar^2 + \ldots + ar^{n-1}$, where a is the first term, n is the number of terms, and r is the common ratio between consecutive terms. This can be called S_n, the sum of the n terms of the geometric series. Because of our theorem, we know $\dfrac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \ldots + x + 1$. So, $a + ar + ar^2 + \ldots + ar^{n-1} = a(1 + r + r^2 + \ldots + r^{n-1}) = a\,\dfrac{r^n - 1}{r - 1}$ or $a\,\dfrac{1 - r^n}{1 - r}$. This may be too abstract for some students to follow. Another way to derive the formula is to first look at $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$ and then to consider $rS_n = ar + ar^2 + \ldots + ar^n$. Subtracting $rS_n - S_n$ leaves $ar^n - a$. Then, $S_n(1 - r) = a(r^n - 1)$, which then gives us $S_n = a\dfrac{1 - r^n}{1 - r}$.

What the TEACHER does:

- Asks students to write a short geometric series and find its sum, such as $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81}$. This should not be converted to decimal form, as accuracy will be lost with rounding. Set up problems that can connect to one of the two methods of deriving the sum formula, such as having students look at $1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81}$ and $\dfrac{1}{3}\left(1 + \dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \dfrac{1}{81}\right)$ and then subtracting the two series to get an expression that is more easily computed and that can be compared with the sum to look for a rule that may arise from the situation.
- Provides contextual examples that use the sum of a geometric series. For example, a building may have 256 windows on the first floor, then 128 on the next floor, up to having 4 on the top floor. How many windows are in the building?

What the STUDENTS do:

- Explain the parts of a geometric sequence and a geometric series.
- Describe how to derive the formula for the sum of a geometric series in their own words.
- Apply the sum for a geometric series to contextual problems.

Addressing Student Misconceptions and Common Errors

Students need to have familiarity with geometric sequences. Work with the standards in F.LE provides students with the connections between exponential growth and geometric sequences, as well as ensuring students know what the parts of a term, ar^n, mean. Using geometric series with few terms and having the students find the sum is an important step. All students should be able to find a sum such as $1 + 2 + 4 + 8$, which will be a foundation on which to build understanding for more complex series, such as those with an r value of 1.012, as may occur in an application. A process by which students can see the steps in creating the formula (as detailed in the "What the Teacher Does" section) helps students make sense of what the formula states, instead of students trying to memorize the formula without any foundational work.

Related Content Standards
F.LE.A.2

Algebra

Domain: Seeing Structure in Expressions
Cluster B: Write expressions in equivalent forms to solve problems.

Standard:

A.SSE.B.3: Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

a. Factor a quadratic expression to reveal the zeros of the function it defines.
b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

Standards for Mathematical Practice:

SFMP 1. Make sense of problems and persevere in solving them.
   Students use factoring, completing the square, graphing, and/or the quadratic formula to solve problems. This requires them to make sense of what the problem is asking and to choose the best method for themselves.

SFMP 2. Use quantitative reasoning.
   Students reason about shapes of graphs, extrema, and zeros. Students compare answers. They justify their steps and listen to what others say for understanding.

SFMP 6. Attend to precision.
   Students use correct vocabulary to share their thinking about the reasoning in the task.

SFMP 7. Look for and make use of structure.
   Students use the structure of quadratic functions to rewrite them and find key components of the graphs.

Goals:

Students use factoring, completing the square, graphing, and/or the quadratic formula to solve problems.

Planning:

Materials: A copy of the "Seeing Structure in Expression" task for students to investigate (**Reproducible 3**).

Differentiating Instruction:

Struggling Students:
Students may need to use a pictorial model to show completing the square.
Students may use a graphing calculator to check their work and their understanding.
A CAS may be helpful to compare steps in a solution.

Extensions:
Students who understand the connections among the representations should explain them to struggling students.

Reflection Questions: Seeing Structure in Expressions

1. What may the different representations of an expression tell a student? For example, why would a student use standard linear form, point–slope form, or slope–intercept form for a linear function?

2. How will you connect factoring, completing the square, and graphing as ways to investigate quadratic functions and higher degree polynomials?

3. How will you incorporate technology into your formative and summative assessments? How can you ensure that students are using appropriate tools strategically?