Part 5

Geometry

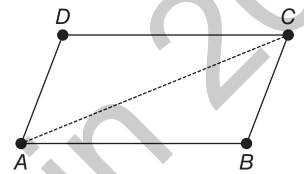
Conceptual Category Overview

The Geometry Conceptual Category builds from the geometric experiences students had at the middle school level, following a trajectory that builds from more hands-on experiences to experiences with formal proof, summarized as follows:

- **Level A.** This level is experiential and overlaps substantially with the Middle Grades Standards.
- **Level B.** This level formalizes the work that has been done at Level A.
- **Level C.** This level serves as a transition to more formal proof, building on transformations as the foundation.
- **Level D.** Formal proof

This progression follows within and across several domains of the Geometry conceptual category, although other domains do not directly address the higher levels of the progression. Note that these levels are somewhat reminiscent of research on the Van Hiele levels, which chart a similar progression from more visual experiences to more formal deduction. Also note that teachers may cover more than one of these levels in the same course, depending on how the high school curriculum is organized. However, this progression may help teachers to think about how the standards within a particular course may be ordered. For example, rather than addressing all of the standards within a domain in one unit, a teacher may choose to roughly order the content by its developmental level.

It is important to note that Level D asks students to be able to write formal proofs. Teachers emphasize having students express their reasoning behind why the theorems work. Allowing students to make conjectures that they then seek to prove or disprove may add both meaning and motivation for writing proofs. However, the standards do not specify the format in which those proofs should appear, and students' reasoning can be expressed in various forms. Students may find explaining their thinking in prose the most accessible since this approach is more open ended. Two-column proofs, in which students place their statements in one column and their reasons for those statements in a second column, are traditionally used in high school. Below, find an example of each format proving that the opposite sides of parallelogram *ABCD* are congruent; this assumes students are familiar with parallel lines and transversals.



Prose

ΔABC and ΔCDA are congruent using the ASA congruence principle. This works since $\overline{AC}$ is shared by the two triangles, and we have two pairs of corresponding congruent angles: $\angle ACB$ and $\angle CAD$, and $\angle CAB$ and $\angle ACD$. We know the angles are congruent since the opposite sides of a parallelogram are parallel, and they are alternating interior angles. Since the triangles are congruent, all of their corresponding parts are congruent, so $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$.

Two-Column Proof

Statement	Reason
ABCD is a parallelogram.	Given
$\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$	Definition of parallelogram
$\angle ACB \cong \angle CAD$ and $\angle CAB \cong \angle ACD$	Alternate interior angles of parallel lines are congruent.
$\Delta ABC \cong \Delta CDA$	SAS Triangle Congruency
$\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{AD}$.	Corresponding parts of congruent triangles are congruent.

Connections are also an important part of this conceptual category, including connections to algebra through coordinate geometry. In addition, there are many opportunities to connect ideas in this conceptual category to the real world through modeling. Students need to be provided opportunities to engage in extended explorations of the modeling cycle at opportune moments throughout their study of the conceptual category, rather than focusing on direct "applications" of the ideas to the ideas being learned.

Direct Connections to Geometry in the Middle Grades

As stated before, the initial standards at the high school level overlap substantially, build upon, and formalize the geometry standards found at the middle grades.

SUGGESTED MATERIALS

Both hands-on materials and various software are important for this conceptual category to support students' exploration of geometric relationship.

G.CO	G.SRT	G.C	G.GPE	G.GMD	G.MG	
✓	✓	✓	✓			Graph paper, tracing paper, patty paper
✓	✓					Mirrors or transparent reflective devices
✓	✓	✓	✓		✓	Geometric tools such as protractors, rulers, and compasses
✓	✓	✓	✓	✓	✓	Dynamic geometry software, such as CoreMathTools, Geometer's Sketchpad, or Geogebra
			✓		✓	Graph paper
				✓	✓	Geometric models of solids
				✓	✓	Three-dimensional modeling software, such as Geogebra or SketchUp

GEOMETRY—OVERARCHING KEY VOCABULARY

G.CO	G.SRT	G.C	G.GPE	G.GMD	G.MG	
✓		✓				**Circumscribed circle for a figure** – When the vertices of a figure lie on the circle. The figure is said to be **inscribed** in the circle.
✓		✓				**Circumscribed figure about a circle** – When the sides of a figure are tangent to the circle. A **circumscribed angle** has sides tangent to the circle.
✓		✓	✓	✓		**Circle** – A set of points in a plane equidistant from a given point, which is called the **center**. The fixed distance is called the **radius**.
✓	✓	✓	✓			**Congruent figures** – One figure is the image of the other using a rigid motion. In congruent polygons, corresponding angles are congruent, as are corresponding sides.
✓	✓	✓	✓			**Corresponding parts** – The sides, angle, and vertices of one figure that are mapped onto those of another figure using a geometric transformation.

GEOMETRY—OVERARCHING KEY VOCABULARY

G.CO	G.SRT	G.C	G.GPE	G.GMD	G.MG	
✓	✓	✓	✓			**Criteria for triangle congruence** – Sets of conditions for determining whether two triangles are congruent. **SSS (side-side-side) triangle congruence** states that if the three pairs of corresponding sides of two triangles are congruent, then the triangles are congruent. **SAS (side-angle-side) triangle congruence** states that if two pairs of corresponding sides of two triangles are congruent, along with the pair of angles included between the two sides, then the triangles are congruent. **ASA (angle-side-angle) triangle congruence** states that if two pairs of corresponding angles of two triangles are congruent, along with the pair of sides included between the two angles, then the triangles are congruent.
	✓	✓	✓			**Criteria for triangle similarity** – Sets of conditions for determining whether two triangles are similar. **AA (angle-angle) triangle similarity** states that if two pairs of corresponding angles of two triangles are congruent, then the triangles are similar.
✓	✓	✓	✓	✓	✓	**Definition** – A description of a mathematical object using known vocabulary.
	✓					**Dilation** – A geometric transformation that preserves angle measurements and the proportionality of side lengths. A dilation is a radial expansion of a figure from a center using a given scale factor. related phrase: *similarity transformation*
✓		✓	✓			**Geometric construction** – A method for drawing a geometric figure without using any pre-existing measurement devices.
✓	✓	✓	✓	✓	✓	**Geometric transformation** – A function that maps all the points of the plane onto the plane. More informally, this may be thought of as a rule for moving or changing a shape. related word: *motion*
✓	✓	✓	✓			**Image** – A figure or set of points that results from a transformation.
✓		✓				**Inscribed circle for a figure** – When the sides of a figure are tangent to a circle. The figure is said to be **circumscribed** about the circle.
✓		✓				**Inscribed figure in a circle** – When the vertices of a figure lie on a circle. The vertex of an **inscribed angle** lies on the circle, and the two sides of the angle intersect the circle.
✓	✓	✓	✓			**Pre-image** – A figure or set of points that is an input to a transformation.
✓						**Rigid motion** – A geometric transformation that preserves length and angle measurements. Examples include rotations, reflections, and translations, as well as sequences of those motions. synonyms: *congruence motion*, *isometry*

GEOMETRY—OVERARCHING KEY VOCABULARY

G.CO	G.SRT	G.C	G.GPE	G.GMD	G.MG	
	✓	✓				**Similar figures** – When one figure is the image of another figure using a similarity transformation. In polygons, corresponding angles are congruent, and ratios of corresponding sides are proportional.
	✓	✓				**Similarity transformation** – A geometric transformation that is a dilation or a sequence of dilations and rigid motions. synonym: *similarity motion*
	✓	✓	✓			**Theorem** – A geometric statement that can be proven to be true based on definitions, axioms, and previously proven theorems.

Congruence (G.CO)

Domain Overview

The Congruence domain builds on geometric transformations, an approach that is quite different from the approach historically taken in high school geometry. The study of congruence follows a trajectory that builds from informal experiences with rigid motions in the middle grades to formal proofs involving criteria for congruence, as illustrated in the following chart.

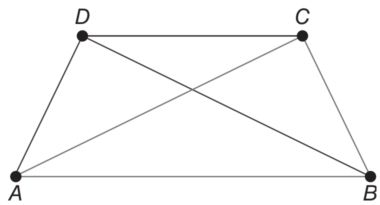
Level A	Level B	Level C	Level D
Carry out transformations in the plane using various tools	Develop definitions of rigid motions in terms of geometric properties. Define congruence using rigid motions.	Develop criteria for triangle congruence in terms of rigid motions.	Prove theorems using congruent triangles.

In the initial Level A, students build on informal experiences from middle grades with reflections, rotations, and translations (also known as rigid motions)—becoming familiar with what they are, how to perform them with a variety of tools and materials, and what their properties are.

In Level B, students subsequently build more formal definitions of the transformations using geometric properties. For example, if point P' is the image of point P when reflected over line m, then $\overline{PP'}$ will be perpendicular to m. Moreover, m will bisect $\overline{PP'}$. The transformations become the basis for defining congruence of two figures in Level C: they are congruent if the one figure results from the other using a sequence of rigid motions.

This leads to an exploration of when two figures will be congruent in Level C, based on just its side and angle measures; if the corresponding angles and sides are congruent, then the figures must be congruent. This then leads to an exploration of criteria for triangle congruence. For example, if the corresponding sides of two triangles are congruent, then there must be a rigid motion relating the two triangles, meaning they are congruent; there is no need to check the angle measures.

In the final level, Level D, this work becomes the basis for students' development of more formal geometric proofs of theorems. For example, we can use triangle congruence to prove that the diagonals of an isosceles trapezoid are congruent. In the figure below, the diagonals of isosceles trapezoid $ABCD$ create two overlapping triangles ΔABC and ΔBAD. Noting that these triangles are congruent using the SAS Triangle Congruence (e.g., $AD = BC$; $\angle BAD \cong \angle ABC$; and $AB = AB$), we can conclude that the corresponding parts $\overline{AC}$ and $\overline{BD}$ must be congruent.



Note that these levels show the progression of the standards within the domain, even if those objectives may all be covered in the same course.

G.CO—KEY VOCABULARY

CO.A	CO.B	CO.C	CO.D	
✓				**Circle** – A set of points in a plane equidistant from a given point, which is called the **center**. The fixed distance is called the **radius**.
	✓			**Criteria for triangle congruence** – Sets of conditions for determining whether two triangles are congruent. **SSS (side-side-side) triangle congruence** states that if the three pairs of corresponding sides of two triangles are congruent, then the triangles are congruent. **SAS (side-angle-side) triangle congruence** states that if two pairs of corresponding sides of two triangles are congruent, along with the pair of angles included between the two sides, then the triangles are congruent. **ASA (angle-side-angle) triangle congruence** states that if two pairs of corresponding angles of two triangles are congruent, along with the pair of sides included between the two angles, then the triangles are congruent.
	✓	✓	✓	**Congruent** figures – One figure is the image of the other using a rigid motion.
✓	✓	✓	✓	**Corresponding parts** – The sides, angle, and vertices of one figure that are mapped onto those of another figure using a geometric transformation.
✓				**Definition** – A description of a mathematical object using known vocabulary.
			✓	**Geometric construction** – A method for drawing a geometric figure without using any pre-existing measurement devices.
✓	✓			**Geometric transformation** – A function that maps all of the points of the plane onto the plane. More informally, this may be thought of as a rule for moving or changing a shape. related words: *motion*
✓	✓	✓		**Rigid motion** – A geometric transformation that preserves length and angle measurements. Examples include rotations, reflections, and translations, as well as sequences of those motions. related words: *congruence motion*, *isometry*
✓				**Symmetric** figure – A figure that has a symmetry. If the symmetry is a reflection, it is **reflectionally (or line) symmetric**. If the symmetry is a rotation, it is **rotationally symmetric.**
✓				**Symmetry** of a figure – A rigid motion in which the figure is mapped onto itself. It is a non-trivial congruence of a figure with itself; that is, the sides and angles do not correspond with themselves.
			✓	**Polygon inscribed in a circle** – A polygon whose vertices all lie on a given circle.
		✓		**Theorem** – A geometric statement that can be proven to be true based on definitions, axioms, and previously proven theorems.
✓				**Undefined notions or terms** – Ideas for which a precise definition cannot be given.

Geometry | Congruence
G.CO.A

Cluster A

Experiment with transformations in the plane.

STANDARD 1 **G.CO.A.1:** Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

STANDARD 2 **G.CO.A.2:** Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

STANDARD 3 **G.CO.A.3:** Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

STANDARD 4 **G.CO.A.4:** Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

STANDARD 5 **G.CO.A.5:** Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Cluster A: Experiment with transformations in the plane.

This cluster begins with Level A, described previously, in which students become familiar with geometric transformations and how to perform them using a variety of hands-on materials and software. They identify that some transformations are "rigid motions" that preserve lengths and angles. They also apply what they are learning to symmetry, which is self-congruence.

At Level B, students revisit the basic vocabulary of geometry and use those terms to define rotations, translations, and reflections. This is built on continuing investigations with hands-on materials and technology.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Students must be challenged to develop deep understanding of the ideas in the cluster through exploring tasks that require problem solving. Students should use a variety of tools, including graph paper, tracing paper, or geometry software. They should look for patterns in their explorations, leading them to making generalizations about the transformations. They should be encouraged to describe their thinking using precise language, to form arguments explaining why certain patterns hold, and to critique arguments that are presented to them. They can use the ideas in this cluster to model real-world contexts.

Related Content Standards

N.VM.C.11 N.VM.C.12 F.BF.B.3 7.G.A.2 8.G.A.1

STANDARD 1 (G.CO.A.1)

Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.

Students will review basic vocabulary of geometry but perhaps with more precision than what they have used in the past. It will be helpful for students to develop and record their own definitions, rather than merely copying definitions from the book.

Also, note that it may be preferable to engage students with this standard *after* they have had basic experiences with rigid motions, which can provide insights into how terms might be defined. For example, describing an angle as a ray along with its image when rotated around its endpoint provides more insight into what an angle is, in conjunction with describing it as the union of two rays at their endpoint. Finally, students need to recognize that while many terms can be defined using known terms, others cannot and serve as basic building blocks for definitions. Such words are called undefined notions or terms.

Finally, note that while not part of the overall development of transformations in this domain, this may be characterized as Level B in that students are formalizing their previous knowledge.

What the TEACHER does:

- Asks students to provide definitions for basic geometric terms, challenging them to be as precise as possible.
- Asks students to make connections with rigid motions in order to better understand the definitions.

What the STUDENTS do:

- Challenge existing ideas of basic geometric terms in order to provide more precise definitions.
- Build understanding of formal definitions for later use in geometric proof.

Addressing Student Misconceptions and Common Errors

Students may have intuitive but limited ideas of basic geometric terms. For example, they may think of a point as a "dot." Teachers should challenge those notions—for example, how big should the dot be? What if the dot is really large? Students should come to recognize that the dot is just a representation of a point, and the point is an exact location (perhaps thought of as the center of the point). They may draw a line as a segment with arrows on the end but not recognize that the arrows are merely a depiction that the line continues in both directions. Even though both point and line are undefined terms, students need to have robust conceptions of what they are.

Students may think of a circle as being "round," which is not precise. To get at the geometric definition, the teacher may ask the students how they could make a circle using transformations; that is, rotate a point all the way around a given point. Likewise, students frequently think of the angle of a figure as the "point" rather than the relationship of two sides. Using rotations to think about how the two sides are related provides a view of an angle based on their relationship and provides a foundation for looking at them relationally.

Connections to Modeling

Geometric objects are abstractions of real-world situations. For example, lines in the real world will not be exactly straight, nor will they go on forever. However, working with a theoretically straight line allows us to formulate models that allow us come to draw conclusions. Since the geometric model is an abstraction that overlooks certain imperfections, care must be taken in interpreting the geometric conclusions reached in accordance with the "Validate" stage of the modeling cycle.

Related Content Standards

G.CO.D G.SRT.A.1 G.C.A.1 G.GPE.A.1

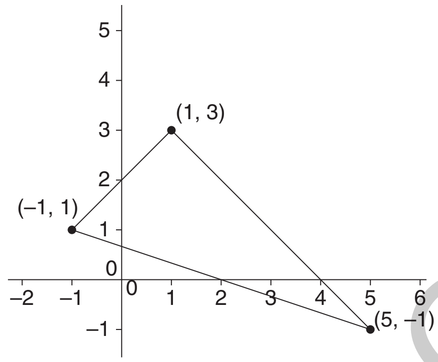
Notes

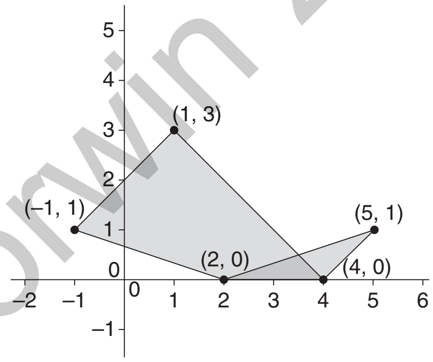
STANDARD 2 (G.CO.A.2)

Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

Students should develop a general understanding of what a transformation is, exploring a range of both rigid (e.g., those that preserve distances and angles) and non-rigid motions (e.g., those that do not). Using *pre-image* to describe an object before being transformed and *image* to describe its result after a transformation can help to simplify the discussion. (Note that these terms are not specified in the standards.) Common notation is to label the image of a point using "prime" notation—as in h' or P', read as *h prime* or *P prime*.

While not specifically included in this standard, use of coordinates can be a useful way to explore and describe a range of transformations. For example, given the first triangle that follows, find its image when doing the following to the *y*-coordinate of each point: (a) add the same value; (b) double it; (c) take its opposite; and (d) take its absolute value. While (a) and (c) preserve lengths and angles, (b) and (d) do not. Students should use the term *rigid motion* to describe (a) and (c). Also, students should be encouraged to consider the images of other points, in addition to the vertices, in order to best understand the effect of the transformation. For example, in (d), the triangle is "folded" along the *x*-axis, as shown in the second figure. This is different from parts (a)–(c) since the function is not one to one; that is, more than one point is mapped to the same image. (This is technically not a transformation, since transformations must be one to one.)





Note that this standard is primarily at Level A, in which students are exploring a range of transformations.

What the TEACHER does:

- Provides students with a wide variety of examples of transformations.
- Provides students with tools for carrying out transformations, including tracing paper, transparent sheets, and geometric software.
- Emphasizes that a transformation is a function.

What the STUDENTS do:

- Carry out transformations of different kinds using a variety of tools, and compare and contrast their effects.
- Recognize that transformations are functions of the plane.
- Recognize that some transformations preserve distance and angle measure while others do not.

Addressing Student Misconceptions and Common Errors

Students often view transformations as a "motion." Teachers should encourage the students to think of a transformation as a function. That is, it is a rule that could be applied to any point in the plane, not just a given figure. To encourage this perspective, teachers may provide a pre-image and image, ask students to describe the transformation, and then use that transformation to find images of additional points or objects.

Students may think that functions only relate to numbers. A teacher might ask students what the definition of a function is and how that might relate to transformations—that is, what would the "input" and "output" of the function be?

Students often confuse transformations of symmetric polygons on the coordinate plane. Teachers can help reduce this confusion and highlight discrepancies in incorrect transformations by ensuring students label points that have been transformed using the prime notation and use manipulatives such as geometric reflectors and patty paper to compare the pre-image to the image.

Connections to Modeling

Students should be encouraged to look for real-world situations that can be thought of as transformations in order to see their relevance and importance. For example, depending on its design, the image of an object in a mirror may produce an image that may appear taller or shorter than the pre-image, or it may be distorted as in a "fun house" mirror.

Related Content Standards

G.CO.B N.VM.C.11 8.G.A.1 8.G.A.3

Notes

STANDARD 3 (G.CO.A.3)

Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.

Students may be familiar with line symmetry but may not relate this idea to transformations. They should see a symmetry as a transformation (namely, a reflection or rotation) that maps a figure onto itself so that the image of every point on the pre-image maps back onto the pre-image. Figures that have a reflection that is a symmetry are reflectionally (or line) symmetric. Figures that have a rotation that is a symmetry are rotationally symmetric.

Note that the symmetries of a figure are useful in understanding the properties of that figure. For example, since a parallelogram has 180° rotational symmetry, its opposite sides and angles will coincide when rotated 180°, meaning they must be congruent. Indeed, characterizing classes of figures in terms of their symmetries can also be useful in understanding how those classes are related. For example, a parallelogram has 180° rotational symmetry. Rectangles, rhombi, and squares all have 180° rotational symmetry (as well as reflectional symmetry), which means they are special kinds of parallelograms. Rhombi have lines of symmetry along their diagonals, and rectangles have lines of symmetry connecting midpoints of opposite sides, which suggests that rhombi and rectangles are not directly related. However, squares have lines of symmetry both along their diagonals and connecting midpoints of their sides, which suggests they are a special case of both rhombi and rectangles.

What the TEACHER does:

- Asks students to find the image of figures using a symmetry.
- Asks students to find all symmetries for different geometric figures or to produce geometric figures with particular symmetries.
- Asks students to explain the properties of a figure based on its symmetries.

What the STUDENTS do:

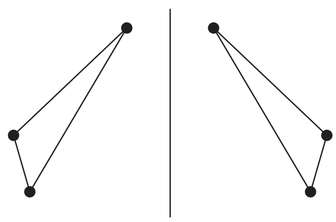
- Understand symmetry in terms of transformations.
- Explore which shapes are symmetric and what symmetries they will have.
- Develop generalizations for the symmetries held by various geometric shapes.
- Determine the properties of a shape based on its symmetries.

Note that this standard is primarily at Level A, in which students are exploring the symmetries of figures using transformations.

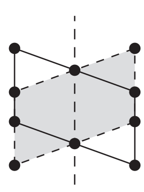
Addressing Student Misconceptions and Common Errors

Students may have problems visualizing symmetry. Using physical materials to explore may be helpful. For example, using a mirror or other reflective surface can help students identify lines of symmetry, as can using tracing paper. However, students need to be encouraged to move beyond a conception that "the two sides are the same" to thinking about the reflection that maps the figure onto itself; that is, each side of the figure is mapped onto the other side. Tracing paper can also be useful in exploring rotational symmetry by seeing when the traced figure matches with the original figure when rotated about a given center. Dynamic geometry software can be helpful with both rotational and reflectional symmetry to explore rigid motions that map the figure back on itself.

Students may conclude that all figures are symmetric, since all of the images of the figure are congruent to the pre-images. In fact, the picture of a figure along with an image does look symmetric, as shown. However, the points of the triangle do not map back onto the pre-image. Thus, the figure is not symmetric, although the triangle and its image *taken together* are symmetric! Students should be encouraged to focus on the pre-image alone.



Students may think that a parallelogram has reflectional symmetry since it has two congruent parts. However, if students are asked to actually carry out the reflection, those parts do not match up, as shown.



Connections to Modeling

Students should be encouraged to explore the symmetries of real-world objects. Efforts should be made to go beyond "applications" of these ideas to a fuller consideration of the modeling cycle discussed in the Modeling conceptual category in Chapter 1.

Related Content Standards

G.CO.C.11 N.VM.C.11 F.IF.B.4 F.TF.A.4

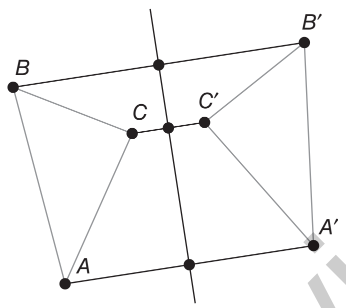
Notes

STANDARD 4 (G.CO.A.4)

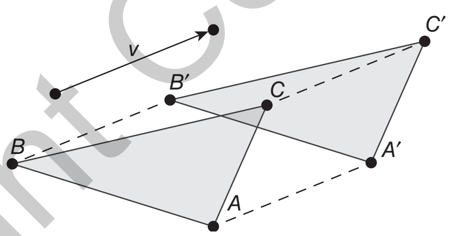
Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

In contrast to the other standards related to transformations in this cluster, this standard focuses on developing more formal definitions of rigid motions and thus is Level B, as described in the introduction to this section. Consequently, teachers should likely develop this standard later in the cluster, after students have had significant experiences with the transformations.

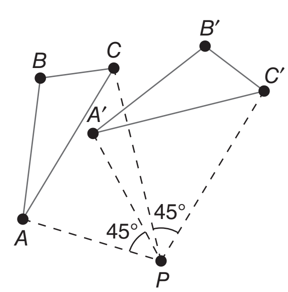
Students are asked to look at the transformation in terms of their geometric properties by looking at how points and their images are related. For example, in a reflection with line l, the segments joining points and their images are perpendicular to the line of reflection. Moreover, l goes through the midpoints of the segments. Thus, we can define a reflection using line l to be a transformation where for each point A, l is the perpendicular bisector of $\overline{AA'}$. SFMP 6 (Attend to Precision) is particularly important with this standard, as we must also account for the case where a point P is on line l, in which case $P = P'$ and there is no segment.



Likewise, in a translation with vector v, the segments joining points and their images are all the same length, corresponding to the length of vector v. Moreover, they are parallel to v, as shown in the following. Thus, we can define a translation using vector v as a transformation in which for each point A, $\overline{AA'}$ is equal in length to v and parallel to v. Again, attention to precision is important, since this definition does not specify the direction of the translation. To account for this, we might add the proviso that the line joining the starting point of v through A and the line joining the ending point of v and A' must be parallel.



Finally, in considering a rotation, each point and its image are the same distance from the center of rotation, which is P in the figure below. Moreover, the angles through the center point from the pre-images to the images are all the same measure. This could again be built into a definition: in a rotation with center P and angle $\propto$, for every point A, $\angle APA'$ will have the same measure as $\propto$. Again, care will be needed to specify the direction of the rotation.



What the TEACHER does:

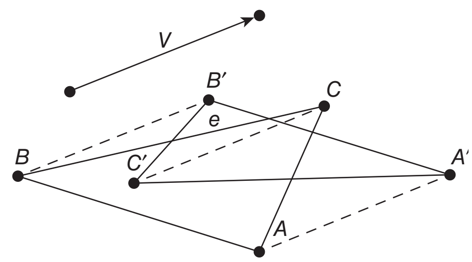
- Asks students to explore the properties of transformations; use of dynamic geometry software may be particularly helpful.
- Ensures precision in the definitions that are developed.

What the STUDENTS do:

- Explore properties of transformations using common geometric relationships (e.g., parallel, perpendicular, and congruence).
- Develop definitions of the transformations in terms of their properties.

Addressing Student Misconceptions and Common Errors

This standard requires a level of precision that may not be comfortable for some students. They may be satisfied in listing the properties of the transformation without noting the limitations of that list in defining the transformation. That is, a transformation may have all of a set of properties, but those properties may not be sufficient to ensure that it is a particular transformation. Teachers may help with this by providing counter-examples. For example, students who note that the lengths of the segments joining points and their images are the same length and parallel then suggest this as a definition for a translation. However, the triangles in the following figure meet those conditions, although they are not related by a translation. The definition needs to also account for the direction of the translation.



Related Content Standards

CO.B.6 8.G.A.1

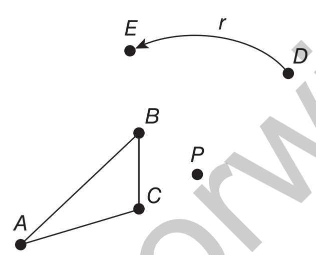
Notes

STANDARD 5 (G.CO.A.5)

Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

This standard is foundational to this cluster and indeed domain, as most of the remaining standards are built on the experiences that students have carrying out rigid motions using a variety of tools. This is clearly a Level A standard. Students may initially be asked to directly carry out the transformation. For example, they may be given a figure and asked to find its reflection using a given line. They can find the image by copying the figure and the line onto tracing paper, folding along the line, and then tracing the image of the figure; they could also place a mirror (or transparent reflective surface) on the line of reflection and draw the image that they see in the mirror.

For a translation, teachers provide students with a figure and a translation vector. If they copy the figure onto tracing paper, they can then move the paper following the vector and trace its image; however, they need to be aware of the direction in which the vector is pointing to be sure their image is in the proper location. For a rotation, teachers provide students with a figure and a center of rotation and either an angle or an angle measure, which the students can trace or draw. Again, the direction of rotation will be a concern; while the convention in mathematics is to rotate counterclockwise unless specified otherwise, teachers may want to explicitly give a direction. One alternative is to use a curved "rotation arrow" showing the direction, as in the following. Polar graph paper may also be useful, where the center of rotation is traced at the origin, and the lines on the paper can then be used to identify angles and distances.



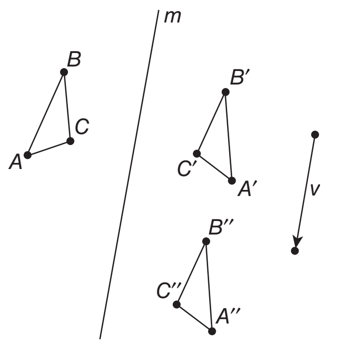
As students become comfortable doing the transformations by hand, teachers can provide them with dynamic geometry software, which will both increase their speed and accuracy as they engage in further and deeper explorations. Given points using coordinates, transformations can then be described in terms of coordinates; for example, "add 2 to each x-coordinate, and add 3 to each y-coordinate" can be used to describe a translation. While important to connect transformations to coordinates, it should not substitute for direct experiences in carrying out the transformations.

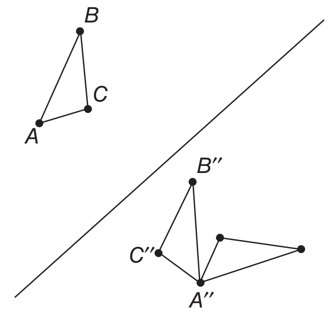
Teachers should also ask students to carry out sequences of transformations. Given that transformations are functions, this is actually a composition of functions, in which the output from one transformation becomes the input for the next transformation. For notation, successive *primes* are frequently added, so that the image of P' is P'' (read *double prime*), and so forth.

As students find the image of a figure using a sequence of motions, a common question is, "Could you find a single rotation, reflection, or translation that will produce this same final image?" This can be an area of rich exploration, as students can discover patterns such as these:

- A sequence of two translations results in a single translation.
- A sequence of two reflections across intersecting lines results in a rotation with the center of rotation at the intersection of the two lines.
- A sequence of two reflections across parallel lines results in a translation.
- A sequence of two rotations will generally be a rotation, except if the sum of their angles of rotation is a multiple of 360°.

However, students should also recognize that it is not always possible to find a single transformation when a translation or reflection is composed with a reflection; the cases in which this occurs are called *glide reflections*. In the first figure, $\Delta A''B''C''$ is the image of ΔABC using a composition of reflection over line m, followed by translation using vector v. $\Delta A''B''C''$ cannot be either a single rotation or reflection of ΔABC, since its orientation is "flipped." Moreover, the line of reflection that maps A onto A'' does not map the remaining vertices onto their corresponding points; see the second figure.





More generally, students may be given a pre-image and image and asked to find the rigid motion (or a sequence of rigid motions) that map the pre-image onto the image. Analyzing the *fixed points* (that is, points whose images are themselves) of a transformation, as well as its effect on the orientation of the figure (whether or not it is "flipped over"), will prove useful in understanding how the rigid motions combine.

What the TEACHER does:

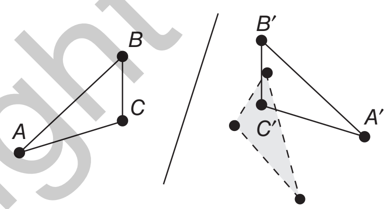
- Provides a variety of experiences for students to carry out the rigid motions, using a variety of tools, including sequences of rigid motions.
- Asks students to find the rigid motion (or sequence of rigid motions) needed to map a pre-image to a particular image.

What the STUDENTS do:

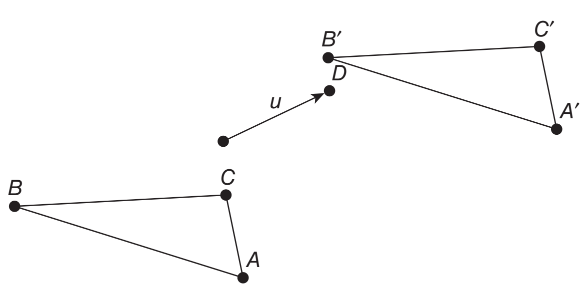
- Develop accurate methods for carrying out the rigid motions using a variety of tools.
- Look for patterns in what happens when carrying out sequences of rigid motions.
- Find rigid motions needed to map a pre-image to a particular image.

Addressing Student Misconceptions and Common Errors

Students may have significant challenges even at the most basic level of carrying out transformations. They may have problems visualizing the effects of a transformation if it is not in a familiar form. For example, while students are most familiar and comfortable working with horizontal and vertical lines, they may struggle when asked to reflect (or translate) along an oblique direction. A typical example of a misconstructed reflection follows, where the student drew $\Delta A'B'C'$ as the image of ΔABC, assuming that the line is vertical; the dotted figure shows the correct image using the oblique line.



Additional problems come when a reflection line, translation vector, or center of rotation overlaps a figure, which may distract students from what the transformation is asking them to do. Likewise, the positioning of a vector can be misleading, as in the following; note that the student has translated too far. Translating or rotating in the right direction can also be a challenge. In all cases, having students focus on specifically carrying out the transformation may be helpful. They could also be asked to use dynamic geometry to check their work and then be asked to explain any discrepancies.



The idea of a sequence of transformations can provide difficulties, as students may find images of the same pre-image rather than using the image of one transformation as input to the next transformation. It is crucial to emphasize how the process works.

Finally, while finding a rigid motion relating a figure to its image using a reflection or translation is fairly straightforward, rotations are much more difficult. Trial and error is initially the only possible approach, and this can be a very time-consuming process. More advanced methods (such as finding the intersection of the perpendicular bisectors of segments joining points and their images) can be explored.

Connections to Modeling

Students should be encouraged to create models of real-world situations using rigid motions in order to see their relevance and importance.

Related Content Standards

G.CO.D 8.G.A.1

Notes

Geometry | Congruence
G.CO.B

Cluster B

Understand congruence in terms of rigid motions.

STANDARD 6 **G.CO.B.6:** Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

STANDARD 7 **G.CO.B.7:** Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

STANDARD 8 **G.CO.B.8:** Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Cluster B: Understand congruence in terms of rigid motions.

This cluster picks up with Level B (developing definitions) to build a definition of triangle congruence in terms of rigid motions. It then moves onto to Level C, developing criteria for triangle congruence to further students' understanding of rigid motions and their relationship to congruence. This builds a foundation for later work with proofs in the next cluster.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.

Students must be challenged to develop deep understanding of the ideas in the cluster through exploring problems that require creating viable arguments and critiquing the reasoning of others. The use of a variety of tools will be useful in making generalizations about when two triangles will be congruent. The need for precision continues to be a major concern for this cluster, as students must thoroughly explain the reasoning behind their work.

Related Content Standards

G.CO.A.2 G.CO.A.4 G.CO.A.5 7.G.A.2 8.G.A.2

Notes

STANDARD 6 (G.CO.B.6)

Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

This standard builds on the work of G.CO.A.4, where students developed geometric definitions of rigid motions. Here, they use those definitions to carry out transformations. Rather than a holistic transformation (or motion) of the figure, they need to find the images of individual points using the definition. For example, to find the image of a triangle using a reflection line, they need to draw perpendicular lines through each vertex and then mark off the same distance from each vertex to the line on the other side of the line in order to find their images.

The second part of this standard introduces the definition of congruence in terms of rigid motions. That is, two figures are congruent if there is a rigid motion so that one is the image of the other.

What the TEACHER does:

- Asks students to use the geometric descriptions of the rigid motions to find an image.
- Introduces the idea of congruence in terms of rigid motions.

What the STUDENTS do:

- Accurately find images of figures using rigid motions.
- Identify if two figures are congruent using rigid motions.

Addressing Student Misconceptions and Common Errors

Students commonly think of congruence as "figures with the same shape and size," and they may revert to that notion. While their previous conception is not wrong, the teacher needs to continually emphasize the link of congruence with rigid motions and show that rigid motions do in fact produce "figures with the same shape and size."

Related Content Standards

G.CO.A.2 G.CO.A.4 G.CO.A.5 8.G.A.2

Notes

STANDARD 7 (G.CO.B.7)

Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

The definition of congruence in terms of rigid motions implies that if two triangles are congruent, then there is a rigid motion by which one is the image of the other. But the parts of one triangle are also the images of the parts of the other triangle using that rigid motion, and so the corresponding parts must also be congruent. Given students' experiences with rigid motions, this should not be at all surprising. Students should use the term *corresponding parts* to simplify the discussion of what is happening.

The other direction, however, is more challenging. If the corresponding parts of two triangles are congruent, can we be sure that the triangles as a whole are congruent? While this may seem intuitively obvious, we need to apply the definition of congruence in terms of rigid motions; that is, can we find a rigid motion that will map one triangle to the other? To clarify the question, the teacher might present two quadrilaterals for which corresponding sides are congruent, but the corresponding angles are not congruent, and ask, Do the quadrilaterals have to be congruent? The point is that we can find a rigid motion that maps one side of a quadrilateral to its corresponding side in the other quadrilateral, but that same rigid motion may not work for another pair of sides; a different rigid motion might be needed. In contrast, with triangles, if we find a rigid motion that maps one side of a triangle to its corresponding side in the other triangle, that same rigid motion will map the remaining angles and sides of the triangle to their corresponding parts in the second triangle.

What the TEACHER does:

- Asks students to compare measurements of the sides and angles of triangles that are related by a rigid motion.
- Poses the question of whether triangles that have congruent corresponding parts must be congruent.

What the STUDENTS do:

- Explain, using rigid motions, why, in congruent triangles, corresponding parts must be congruent.
- Explain that if two triangles have congruent corresponding parts, then the triangles must be congruent.

Addressing Student Misconceptions and Common Errors

Students may focus on the sufficient condition in the definition and see less of a point for dealing with the converse, in which we show that triangles are congruent when the condition is met. The prior discussion provides an example of how to motivate that discussion.

Related Content Standards

G.SRT.A.2   8.G.A2

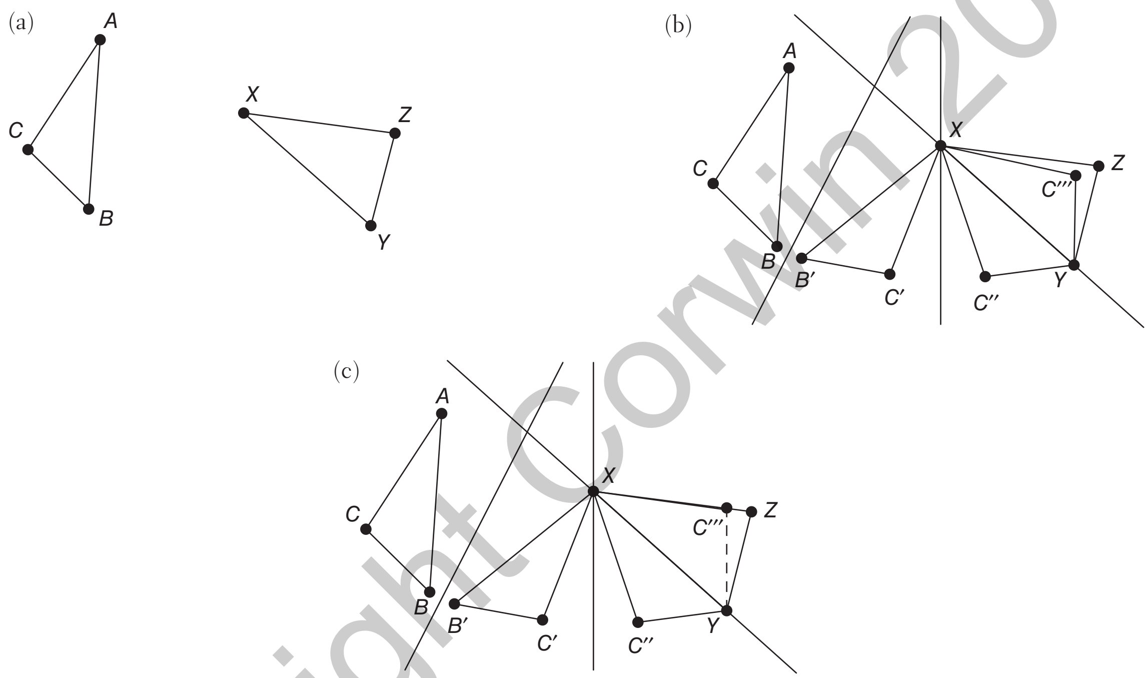
Notes

STANDARD 8 (G.CO.B.8)

Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

This standard naturally builds on the previous standard, in which the focus was on whether we can be sure that two triangles are congruent if their corresponding parts are congruent. Students now consider what happens if some (but not all) of their corresponding parts are congruent. Is it certain that the triangles are congruent? Students might be asked to investigate which pairs of corresponding parts are needed to ensure the triangles are congruent. Again, the definition of congruence in terms of rigid motions is used to show these criteria work; can we find a rigid motion that will map the one triangle onto the other?

For example, consider in ΔABC and ΔXYZ, shown below, $AB = XY$, $\angle A$ is congruent to $\angle X$, and $\angle B$ is congruent to $\angle Y$. Since $AB = XY$, we can find a rigid motion that maps $\overline{AB}$ onto $\overline{XY}$, say, using two reflections, as shown. Will a reflection over $\overline{XY}$ map C'' onto Z? The picture is "rigged" to make this look like it won't work. However, $\angle A$ is congruent to $\angle X$, which means $\angle YXC''$ is congruent to $\angle YXZ$. So $\overrightarrow{XC'''}$ must be the same as $\overrightarrow{XZ}$, as shown in part (c) of the figure. Likewise, $\overrightarrow{YC'''}$ must be the same as $\overrightarrow{YZ}$, which means that C''' must land on Z, and the two triangles are thus congruent.



This demonstrates that if two triangles have two pairs of corresponding congruent angles and if the pair of corresponding sides between the angles is congruent, then the triangles must be congruent. This is commonly referred to as Angle-Side-Angle (ASA) Triangle Congruence. Similar demonstrations can be made for Side-Angle-Side (SAS) and Side-Side-Side (SSS) Triangle Congruence.

What the TEACHER does:

- Presents the students triangles with various sets of congruent parts.
- Asks the students to determine if the triangles must be congruent.

What the STUDENTS do:

- Explore what sets of congruent parts that triangles may have that will ensure that they are congruent.
- Use the definition of congruence to verify their conclusions.

Addressing Student Misconceptions and Common Errors

Many students will likely find the logic behind this standard challenging. They might be given physical models to help them understand that certain sets of corresponding congruent parts will ensure that two triangles are congruent; for example, they might explore the ways that three sticks can be used to form a triangle in order to help them to visualize the relationships. They might also use dynamic geometry software to explore what happens given different sets of congruent corresponding parts in two figures.

Connections to Modeling

Students might explore how these conditions could be used to construct triangles in the real world. For example, what measurements would you need to provide a friend so that he or she could draw a triangle exactly like one you have drawn on your paper?

Related Content Standards

G.SRT.A.3 7.G.A.2

Notes

Geometry | Congruence
G.CO.C

Cluster C

Prove geometric theorems.

STANDARD 9 **G.CO.C.9:** Prove theorems about lines and angles. *Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.*

STANDARD 10 **G.CO.C.10:** Prove theorems about triangles. *Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*

STANDARD 11 **G.CO.C.11:** Prove theorems about parallelograms. *Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.*

Cluster C: Prove geometric theorems.

This cluster is solidly at Level D, as described in the introduction to this conceptual category, and should follow substantial work with the preceding clusters. Moreover, allowing students to make conjectures that they then seek to prove or disprove may add both meaning and motivation for writing proofs. For example, students may explore the measurements of sides and angles in a parallelogram construction in dynamic geometry software or, alternatively, measure the sides and angles in a series of drawings of parallelograms and make conjectures about possible properties parallelograms may have. These observations may then be formalized into conjectures that students seek to prove or disprove the properties.

While these standards ask students to write formal proofs, they do not specify the format in which those proofs should be written. Students should be encouraged to express their reasoning behind why the theorems work. See the introduction to this conceptual category for additional discussion of approaches to helping students build facility with writing proofs.

Note that while the use of transformations is not explicitly required in these standards, students' prior experiences with transformations can inform their thinking as they construct proofs. Depending on the approach taken, teachers may explicitly allow or encourage thinking based on transformations in students' proofs.

Standards for Mathematical Practice
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

The major focus of this cluster is on SFMP 3; in addition to creating proofs (constructing viable arguments), they should have an opportunity to compare their proofs to those created by their classmates (critiquing the reasoning of others). Attention to precision (SFMP 6) is also essential, as students need to use precise mathematical language to express their reasoning. Looking for and making use of structure (SFMP 7), as well as patterns (SFMP 8), are important aspects of conjecturing, and use of appropriate tools may support explorations leading to their making conclusions.

Related Content Standards
G.CO.D G.SRT.B.4 8.G.A.5

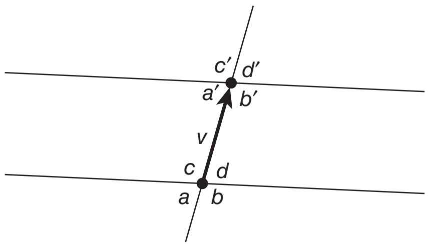
STANDARD 9 (G.CO.C.9)

Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

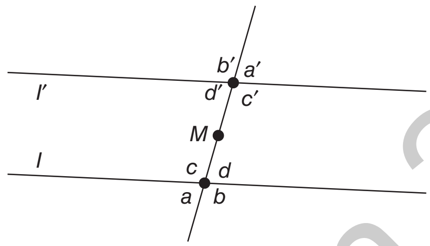
These basic theorems build from basic understandings about lines and angles, such as supplementary angles. For example, vertical angles are congruent because they are supplementary to a common angle.

Rigid motions can be useful in understanding the relationships between angles formed by a transversal; as seen in figure (a) below, translating line *l* using vector *v* results in a parallel line *l*′, where the corresponding angles must be congruent. Vertical angles can then be used to prove that the alternate interiors angles are also congruent. Alternatively, a rotation of 180° about the midpoint of the segment joining the two lines can also be used to show alternate interior angles are congruent; see figure (b).

(a)



(b)



Finally, noting that the perpendicular bisector of a line segment will be a line of reflection mapping one endpoint onto the other makes it clear that the points on that line must be equidistant from the segment's endpoints, since the segments joining a point on that line to the endpoints will be congruent.

What the TEACHER does:

- Asks students to explore situations and make conjectures about situations involving lines and angles.
- Encourages students to prove whether those conjectures are really true, using transformations as a useful way of thinking.
- Promotes precision in students' arguments.
- Provides opportunities for students to critique one another's proofs.

What the STUDENTS do:

- Make and prove conjectures about situations involving lines and angles.
- Express their proofs using precise mathematical language.
- Use transformations as a way about those relationships.
- Examine and critique proofs produced by other students, re-examining their own proofs in light of what other students have done.

Addressing Student Misconceptions and Common Errors

Students frequently have difficulties making the correct correspondence between the angles of parallel lines formed by a transversal; using transformations to understand the relationship between the two lines as either a translation or rotation will help.

Students may have difficulty expressing their thinking in more formal ways. The teacher needs to encourage precision in his or her communication both orally and in written work. Classroom dialogue can also help students see the limitations of their thinking.

Connections to Modeling

Relations among angles formed by parallel lines and the results of other theorems addressed in this standard can be seen in many real-world situations. Those contexts can help to motivate the need for these theorems, and those theorems can, in turn, be used to analyze such situations. For example, when exploring possible locations where a person can stand so that he or she is the same distance from two other (stationary) people, students will see that all such locations will be on the perpendicular bisector of the segment joining the locations of the other two people.

Related Content Standards

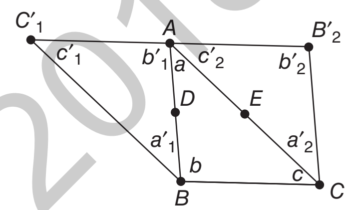
G.GPE.B.5 G.SRT.B.5 8.G.A.5

STANDARD 10 (G.CO.C.10)

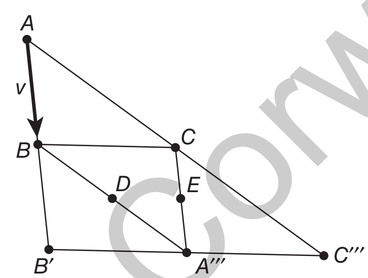
Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

These theorems build on those from the previous standard and should again build on student conjectures about the relationships described in the standards.

While conventional approaches to these proofs might be expected, transformations can again provide powerful insights into what is happening. For example, consider ΔABC along with two images of ΔABC when rotated by 180° around D and E, the midpoints of two of its sides, as shown. Then, $\overline{C'_1A}$ and $\overline{AB_2'}$ are both parallel to $\overline{BC}$ (since rotated by 180°), and so, $\overline{C'_1B'_2}$ includes point A. Thus, $\angle b'_1$, $\angle a$, and $\angle c'_2$ must add up to 180°. Put another way, $\angle b$ and $\angle b'_1$ are alternate interior angles, as are $\angle c$ and $\angle c'_2$, which underlie the more conventional proof.



Likewise, the proof that the base angles of an isosceles triangle are congruent can build on use of congruent triangles. Seeing that the base angles are images of each other when reflected over the perpendicular bisector of the base provides useful insights into what is happening and which triangles need to be proved congruent. And transforming a triangle using a sequence of a translation by vector v and then two rotations about D and E, as shown below, can provide insights into why a midsegment of a triangle is parallel to the third side and half its length. Noting that ΔABC is similar to $\Delta AB'C'''$ also provides a preview into how the areas of similar triangles are related, since four copies of ΔABC fit into $\Delta AB'C'''$.



What the TEACHER does:

- Asks students to explore situations and make conjectures about situations involving triangles.
- Encourages students to prove whether those conjectures are really true, using transformations as a useful way of thinking.
- Promotes precision in students' arguments.
- Provides opportunities for students to critique one another's proofs.

What the STUDENTS do:

- Make and prove conjectures about situations involving triangles.
- Express their proofs using precise mathematical language.
- Use transformations as a way about those relationships.
- Examine and critique proofs produced by other students, re-examining their own proofs in light of what other students have done.

Addressing Student Misconceptions and Common Errors

Students may have difficulty expressing their thinking in more formal ways. The teacher needs to encourage precision in students' communication. Classroom dialogue can also help students see the limitations of their thinking. Having students restate the explanations given by other students can help reduce the teacher's need to critique those explanations and provide opportunities for the teacher to gauge student learning.

Connections to Modeling

These theorems all relate to many real-world situations that students encounter. Such situations can provide both motivation and opportunities for application. For example, students might explore that the centroid of a triangle, where its medians meet, is the balance point for a triangle using a cardboard triangle.

Related Content Standards

G.SRT.B.5 8.G.A.5

STANDARD 11 (G.CO.C.11)

Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

The theorems in this standard build on those addressed in the previous standard, such as the relationships of vertical angles and the angles formed by parallel lines. Again, understanding of the theorems should build on student conjectures with parallelograms and other quadrilaterals. For example, using dynamic geometry or drawings to explore the properties of parallelograms, rectangles, squares, and rhombuses will lead to conjectures about what properties they have and should precede asking students to prove that they are theorems. These explorations also solidify how different classes of quadrilaterals are related. For example, a rectangle has all of the properties of a parallelogram and so can be thought of as a special kind of parallelogram with right angles. Likewise, a square can be thought of as a special kind of rectangle (with equal sides) and a special kind of rhombus (with right angles).

As with the previous standards, these theorems can be proved using congruent triangles. However, students can build on their previous experiences with symmetry. Recognizing that parallelograms have 180° rotational symmetry can guide their thinking about the properties that parallelograms have: The opposite angles are images of each other by the rotation, and since the intersection of the diagonals is the center of rotation, the diagonals are bisected by each other. Since rectangles also have reflectional symmetry, in addition to 180° rotational symmetry, their diagonals not only bisect each other but are also congruent since they are images of each other over the line of symmetry. In fact, rectangles will inherit all of the properties of parallelograms due to the 180° rotational symmetry, another way of seeing they are a special kind of parallelogram.

What the TEACHER does:

- Asks students to explore situations and make conjectures about the properties of a parallelogram.
- Encourages students to prove whether those conjectures are really true, using transformations as a useful way of thinking.
- Promotes precision in students' arguments.
- Provides opportunities for students to critique one another's proofs.

What the STUDENTS do:

- Make and prove conjectures about the properties of a parallelogram.
- Express their proofs using precise mathematical language.
- Use transformations as a way about those relationships.
- Examine and critique proofs produced by other students, re-examining their own proofs in light of what other students have done.

Addressing Student Misconceptions and Common Errors

Students may have difficulties making the correct correspondence between the vertices of the triangles used to prove some of these properties, since they are based on a rotation. However, referring back to transformations in general and symmetry specifically may help them better understand the correspondence. Students may mistakenly believe that the diagonals of a parallelogram are congruent; again, referring back to rotational symmetry may help them to see that one diagonal is not rotated onto the other; instead, each diagonal is rotated onto itself. This shows that the diagonals are not congruent; rather, they bisect each other.

Students may have difficulty expressing their thinking in more formal ways. The teacher needs to encourage precision in their communication. Classroom dialogue can also help students see the limitations of their thinking.

Connections to Modeling

Parallelograms, rectangles, and other quadrilaterals are prevalent in students' real-world experiences, and connections should be made to those experiences.

Students can use sticks to represent the diagonals of a quadrilateral and adjust them to see how they should be arranged to form different types of quadrilaterals, helping them to see the properties of the diagonals of quadrilaterals. This can be framed as a "kite building" project, as at http://www.insidemathematics.org/classroom-videos/public-lessons/9th-10th-grade-math-properties-of-quadrilaterals.

Related Content Standards

G.CO.A.3

Geometry | Congruence
G.CO.D

Cluster D

Make geometric constructions.

STANDARD 12 **G.CO.D.12:** Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). *Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.*

STANDARD 13 **G.CO.D.13:** Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Cluster D: Make geometric constructions.

The emphasis of this cluster extends beyond merely carrying out a series of steps to obtain a given outcome. Rather, students might be asked to describe the steps that they could use. In either case, they should explain why the construction works.

Standards for Mathematical Practice
SFMP 1. Make sense of problems and persevere in solving them.
SFMP 3. Construct viable arguments and critique the reasoning of others.
SFMP 4. Model with mathematics.
SFMP 5. Use appropriate tools strategically.
SFMP 6. Attend to precision.
SFMP 7. Look for and make use of structure.
SFMP 8. Look for and express regularity in repeated reasoning.

Nearly all of the Standards for Mathematical Practice may come into play with this cluster. Most obviously, the use of tools is central to the cluster. Likewise, generating and critiquing arguments will be important in understanding why certain constructions work. In generating their own constructions, students will need to make sense of the problem and persevere, then look for patterns that will help them see general methods that can be used. Finally, attending to precision will be of crucial importance, since even small errors in executing a construction may lead to results that don't work.

Related Content Standards
G.CO.A.1 G.CO.A.5 G.CO.C.9-11 7.G.A.2

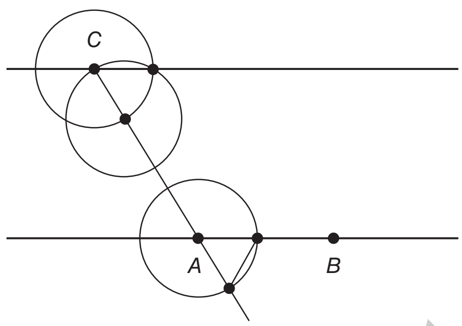
Notes

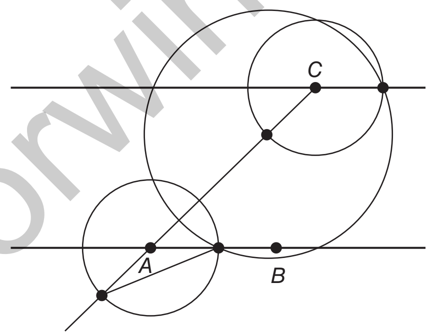
STANDARD 12 (G.CO.D.12)

Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

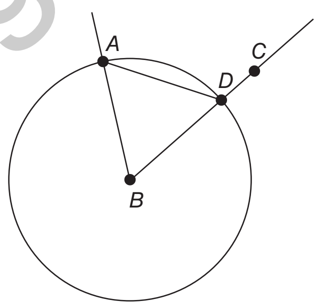
Students should have an opportunity to create the constructions discussed in this standard, rather than merely carrying out a "recipe." A variety of tools should be used to make the constructions, including "compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc." The commonality among these tools is that the use of measurement devices is not permitted. For example, students can replicate a length by setting the radius of a compass equal to a given segment and drawing an arc of that length, marking off the segment on a piece of string or on the edge of a paper and copying it, and using a reflective device or tracing paper (often called "patty paper") to trace an image. In each case, however, students cannot use a ruler to measure the length of a segment using some unit of measurement and then replicate a segment using the same measure. Likewise, they cannot use a protractor.

When using dynamic geometry software, students should not be allowed to use built-in constructions; rather, they should be limited to a virtual compass (circle by center and radius) and the line tool. Moreover, they should not rely on measurements of length or angles to construct the object. Instead, their constructions should be built on relationships among the objects in the construction. If a construction is valid, it should still work when the points in the construction are moved; this is sometimes called the *drag test* since the points are being dragged around the screen. For example, consider the construction of a line parallel to $\overline{AB}$ through point C shown in the figure on the left below. If point C is dragged to a new location, as shown in the figure on the right, the elements of the construction will dynamically adjust, and the line through C will remain parallel to $\overline{AB}$.





Many of these constructions build on each other. For example, the construction for bisecting a segment generally is based on constructing its perpendicular bisector. The perpendicular bisector construction can be extended to many other constructions, such as bisecting an angle and constructing a line that is perpendicular (or parallel) to a given line. Consider, for example, constructing the bisector of an angle ABC, shown below. We draw circle B through point A, intersecting ray BC at point D. Constructing the perpendicular bisector of AD will also bisect the angle.



Having students explain why a construction works is also important. In the previous example, triangle ADB is isosceles, since AB and BD are radii of the same circle. Thus, the perpendicular bisector of its base will be a line of symmetry for the triangle, meaning that line must also bisect the vertex angle of the triangle. A more formal proof using congruent triangles (or a previously proved theorem about isosceles triangles) can also be written.

What the TEACHER does:

- Asks students to develop procedures to construct various objects using a variety of tools.
- Asks students to explain why particular constructions work.

What the STUDENTS do:

- Devise potential methods for constructing various objects.
- Prove that their constructions work.

Addressing Student Misconceptions and Common Errors

Students frequently want to resort to using a ruler or protractor. The teacher needs to make the constraints for use of a particular tool clear.

If students are not precise in a construction, it may not appear to work. The teacher needs to emphasize the importance of precision. Alternatively, using dynamic geometry software could alleviate some of these difficulties.

Connections to Modeling

Students explore how these constructions might be useful in real-world contexts, such as carpentry or landscaping.

Related Content Standards

G.CO.A.1 G.CO.A.5 G.CO.C.9-11 G.C.A.3 G.C.A.4 7.G.A.2

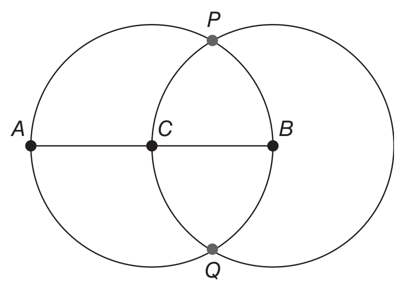
Notes

STANDARD 13 (G.CO.D.13)

Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

This standard expands on the constructions created in the previous standard, as well as on the knowledge built throughout the domain. For example, to construct a square that is inscribed in a circle (that is, all of the square's vertices lie on the circle), a student might observe that the diagonals of a square are perpendicular. Thus, constructing two perpendicular diameters of the circle will determine the four vertices of the square, the points where the diameters intersect the circle.

Likewise, given circle C with diameter $\overline{AB}$, drawing circle B through point C results in two points of intersection labeled P and Q. Noting that $PC = CB = BP$ provides an important clue for constructing an equilateral triangle, as well as for constructing an inscribed hexagon by repeating this construction several times.



More advanced knowledge from other domains (particularly trigonometry) could also be used in showing why some of these constructions work.

What the TEACHER does:

- Asks students to develop procedures to construct various objects using a variety of tools.
- Asks students to explain why particular constructions work.

What the STUDENTS do:

- Devise potential methods for constructing various objects.
- Prove that their constructions work.

Addressing Student Misconceptions and Common Errors

If students are not precise in their construction, it may not appear to work. The teacher needs to emphasize the importance of precision. Alternatively, using dynamic geometry software could alleviate some of these difficulties.

Connections to Modeling

While, in many real-world contexts, use of measurement is common in creating geometric figures, in some situations, the constructions in this standard may be useful. Teachers might have students explore situations where using angle and side measures will be simpler and where using a construction might be simpler. For example, to locate where three trees should be planted so that they are the same distances apart from each other, it will be easier to use ropes of the same length rather than trying to use a tape measure and protractor. Teachers might also have students look at the relative accuracy of the drawings produced by measurement and by construction; in contrast to what might be expected, the constructions are often more accurate.

Related Content Standards

G.CO.A.1 G.CO.A.5 G.CO.C.9-11

Notes

Sample PLANNING PAGE

Geometry

Domain: Congruence

Cluster A: Experiment with transformations in the plane.

Standards:

G.CO.A.4: Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G.CO.A.5: Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

Standards for Mathematical Practice:

SFMP 1. Make sense of problems and persevere in solving them.
Students will reason about what motions could work and realize that it must include a rotation. But finding exact motion(s) that work will require perseverance.

SFMP 3. Construct viable arguments and critique the reasoning of others.
Students need to explain their reasoning behind different motions or sequences of motions that can be used and how they are related.

SFMP 5. Use appropriate tools strategically.
Students can use either a dynamic geometry system or paper and pencil to explore what motions can be used.

SFMP 6. Attend to precision.
While appearances might lead students to certain conclusions, they need to precisely describe the motions that they use and ensure that they work as they thought.

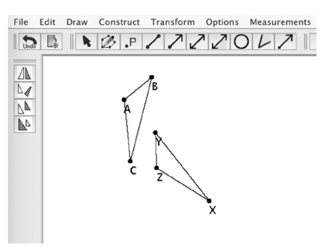
SFMP 7. Look for and make use of structure.
Students need to understand the effect of different motions on an object and how the effects of those motions interact. For example, reflecting an object reverses its orientation, so it will require two (or an even) number of reflections to produce an object with the same orientation.

Goals:

Students have been working with rigid motions. Now, they will consider different sequences of rigid motions that relate two congruent figures. Through this, they explore properties of the rigid motions and form generalizations about how they are related.

Planning:

Materials: Dynamic geometry software with two congruent scalene triangles drawn, as shown. Alternatively, they could be given the task on paper and asked to explore using paper-and-pencil constructions.



Task: Find a sequence of rigid motions that maps one triangle onto the other.

Sample PLANNING PAGE (Continued)

Questions and Prompts:

How can you describe the rigid motions you used?

For example, they may translate B *to* Y, *then rotate about that point using a 45° angle.*

Can you find a different sequence of rigid motions that will work?

Students may see that there are many ways to do this. For example, they could translate any vertex to its corresponding angle, then rotate 45° about that point.

Could you use a single rigid motion? How could you describe that motion?

Students may see they can use a rotation of 45°. However, finding the location of the center may be difficult. They could draw a 45° rotation, then drag the center so that the two figures are superimposed.

Can you use only reflections? How many reflections will you need to use? Why?

Students may not think about using reflections. However, they should see that if you reflect ΔABC *using a line that maps one vertex of* ΔABC *onto its corresponding vertex, you can then find a second intersecting line that will reflect* ΔA′B′C′ *onto the final image,* ΔZYX. *They should also note that it must take two reflections, since a reflection "flips the shape over" (reverses its orientation), so a second reflection will be needed to "flip the shape back."*

How are your reflection lines related to the single motion?

If they measure the angle between the lines, they may notice that it is half the angle of rotation.

They may also notice that the point of intersection of the two reflection lines is the center of rotation! A discussion of the points that are fixed (that is, whose images are themselves) by a reflection may help them to see that the point of intersection of the two lines must be fixed, which makes it the center of rotation.

Differentiating Instruction:

Struggling Students:

Allow time for students to become familiar with the dynamic geometry software so that they can fluently perform the motions rather than use of the software becoming their focus.

For some students, finding a sequence of rigid motions that will work may be a challenge. They should freely explore the situation without feeling pressure to find the "best" solution.

Extensions:

Students who need additional challenge may be encouraged to explore what happens with other sequences of reflections. Will a sequence of two reflections always be a rotation? What happens if the reflection lines are parallel? What happens with a sequence of three reflections?

Notes

Reflection Questions: Congruence

1. What are the advantages to using geometric transformations as the foundation for the study of congruence?

2. How does symmetry build on the use of geometric transformations?

   a. How does the use of transformations help to simplify discussions of symmetry?

   b. The standards include reflectional and rotational symmetry. Is translational symmetry possible? Are there any other possible kinds of symmetry?

3. When students are using dynamic geometry software to make conjectures about a situation, how can they be motivated to prove their conjectures are true rather than merely relying on their observations, which may seem very convincing?

4. Carrying out constructions is often viewed as a fun, "hands-on" activity for students. How can formal constructions be made a meaningful part of the curriculum rather than merely a chance for students to play with compasses and rulers?