

THE LEARNING FRAMEWORK *IN* NUMBER

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THE LEARNING FRAMEWORK *IN* NUMBER

PEDAGOGICAL TOOLS FOR
ASSESSMENT AND INSTRUCTION

ROBERT J. WRIGHT
DAVID ELLEMOR-COLLINS



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About the Authors and the Contributor

Dr Robert J. (Bob) Wright holds Bachelor's and Master's degrees in mathematics from the University of Queensland (Australia) and a doctoral degree in mathematics education from the University of Georgia. He is an adjunct professor in mathematics education at Southern Cross University in New South Wales. Bob is an internationally recognized leader in assessment and instruction relating to children's early arithmetical knowledge and strategies, publishing six books, and many articles and papers in this field. His work over the last 25 years has included the development of the **Mathematics Recovery** Programme, which focuses on providing specialist training for teachers to advance the numeracy levels of young children assessed as low-attainers. In Australia and New Zealand, Ireland, the UK, the USA, Canada, Mexico, South Africa and elsewhere, this programme has been implemented widely, and applied extensively to classroom teaching and to average and able learners as well as low-attainers. Bob has conducted several research projects funded by the Australian Research Council including the most recent project focusing on assessment and intervention in the early arithmetical learning of low-attaining 8- to 10-year-olds.

David Ellemor-Collins holds a Bachelor's degree with honours in mathematics and philosophy from Harvard University, and a Graduate Diploma in Education from the University of Melbourne. David works as a mathematics teacher, as he has done for 20 years across primary schools, high schools and universities. As a specialist in arithmetic instruction, he publishes articles for both researchers and practitioners, designs curriculum materials, and provides professional development to teachers. He has co-authored a book with Bob, and collaborated on Bob's recent research project on intervention with low-attaining 8- to 10-year-olds. David is completing a doctoral degree in mathematics education at Southern Cross University, focusing on instruction in multiplication and division.

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Foreword

In this book, Robert J. Wright and David Ellemor-Collins do not view children's minds as *tabula rasa* upon entering school, nor do they regard children's mathematical learning as simply imprinting conventional school mathematics on children's minds. Rather, they fully acknowledge children's mathematics, which are the mathematical ways of operating that children construct as a result of interacting in their social-cultural milieu in all of its aspects and of maturation of the central nervous system. Acknowledging children's mathematics is a basic principle for any adult who is involved in children's mathematics education simply because children are human beings not unlike adults. With nothing else being said, adults do not need to interact with children in order to impute some kind of mathematics to them; that is, children's mathematics. But adults do need to interact with children in order to construct what children's mathematics might consist of. Throughout this book, the authors celebrate children as mathematical thinkers and learners and fully subscribe to the position that teachers construct what children's mathematics might consist of in the context of teaching children. However, children's mathematics is not simply lodged in teachers' minds through their senses in a way analogous to how a postage stamp is stuck on an envelope. Rather, in the context of teaching children, teachers must interpret children's mathematical language and actions and continually test their interpretations in order to construct experiential models of children's mathematics. Because such experiential models are constructed by teachers it might seem that the models would be isomorphic to teachers' ways and means of operating mathematically. But teachers must become the children and put their own mathematical ways and means of operating 'on the side', as it were, and construct a 'mathematics' – some experiential model of children's mathematics – that, were children to have it, the children would produce the observed mathematical behavior. That is, teachers must learn to think mathematically as if they were the children. In doing so, the goal is for teachers to construct a 'mathematics' that fits within the constraints of children's mathematical language and actions.

Through their own work teaching children, through their research with children and teachers, as well as through their interpretation of the research of others, Wright and Ellemor-Collins have developed a comprehensive *Learning Framework in Number* (LFIN) across the first six grades in school that represents their generalized models of the mathematics of children across those six grades. Their LFIN represents a major step forward toward the inclusion of children's mathematics in mathematics curricula for children. Correlated with age ranges, three broad bands of numeracy are laid out and, within each band, domains of number knowledge are specified and developed. But to regard LFIN simply as a blueprint for mathematics curricula for children would be to drastically short-change it. Also included is a vision of how assessment within the domains of number knowledge can be used to inform mathematics teaching and children's learning. Based on assessment, the authors advocate that teachers pitch instruction just beyond the cutting edge of students' learning. In this way, the tasks that teachers present to children are at a level that makes sense to them in terms of their current knowledge. But to solve the tasks, children would need to reorganize their approach, develop new strategies, or gain new insights.

The LFIN is a model that is finely tuned to children's mathematics in its levels of complexity and sophistication. In the model, children's mathematical behavior constitutes raw material for teachers' interpretative and constructive activity concerning pedagogical decisions. The teaching approach is inquiry-based and children are routinely engaged in 'thinking hard to solve numerical problems'. In this context, it is important to note that the authors leave open the possibility that teachers may well encounter children operating numerically in ways that are not accounted for in their LFIN model. Furthermore, the model does not prescribe what teachers are to do given current assessment results. The authors' understanding of teachers as rational and creative is essential, especially in those cases where children meet situations subsequent to an assessment that their current mathematics is not sufficient to solve. These cases can be particularly challenging because it is children's actions that determine the effectiveness of a planned intervention rather than the intentions of the teacher. If children do not engage in effective action regardless of a teacher's interventions, the creative teacher must 'step back' and work to establish learning trajectories that may not be pre-specified and that occasion children's effective action. The authors' view of teachers as rational and creative especially recommends the LFIN and Mathematics Recovery Program. When coupled with their emphasis on children's mathematics, the program is revolutionary.

Leslie P. Steffe



Acknowledgements

This book is a culmination of several interrelated research projects conducted over the last 25 years, many of which come under the collective label of Mathematics Recovery. All of these projects involved undertaking research, development and implementation in collaboration with teachers, schools and school systems. These projects have received significant support from the participating schools, school systems and jurisdictions.

The authors wish to express their sincere gratitude and appreciation to all of the teachers, students and project colleagues who have participated in, and contributed to, these projects. We also wish to thank the following organizations for funding and supporting one or more projects which have provided a basis for writing this book: the Australian Research Council under Grant No. AM9180064 and Grant No. LP0348932; Southern Cross University; the government and Catholic school systems of the north coast region of New South Wales, Australia; the New South Wales Department of Education and Training; the School District of Oconee County and the South Carolina Department of Education; many other school districts across the United States; the University of Liverpool, and Wigan, Sefton, Salford, Stockport, Knowsley and Cumbria Education Authorities in England; Flintshire County Council in Wales; the Ministry of Education in the Bahamas; Catholic Education Melbourne (Australia); The First Peoples Center for Education (USA); the Department of Education and Science, Ireland; The Kentucky Center for Mathematics; the Frontier School Division, Manitoba, Canada; the University of Strathclyde, and Glasgow, Edinburgh, and Stirling Education Authorities in Scotland; Fundación Educación, Voces y Vuelos, IAP in Mexico; and Trinity College, South Australia.

Finally, we wish to thank the following people for their contributions to the development and refinement of important ideas in this book: Andrea Dineen, Peter Gould, Lucy Kett, Kurt Kinsey, Thi Tran Le, Jim Martland, Petey MacCarty, Joanne Mulligan, Vicki Nally, Rumiati, Ann Stafford, Garry Stanger and Pam Tabor.



Catholic Education Melbourne

A special acknowledgement

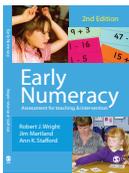
The material in this book is drawn from two major initiatives focusing on intervention in the number learning of low-attaining students in the range from the Foundation Year (Kindergarten/Prep) to 4th Grade. The first of these focused mainly on the Foundation Year to 1st Grade, and the second extended this range to 4th Grade. The second of these two initiatives involved an extended collaboration between the authors of this book and *Catholic Education Melbourne (CEM)*. This collaboration began with a three-year Linkage Research Project (2004–6) funded jointly by the Australian Research Council (under Grant #LP0348932) and CEM. Commencing in 2009, this collaboration has continued each year via CEM's Number Intervention F-4 Project. Each year, throughout this period, the first author has provided an extended programme of professional learning with one or more cohorts of CEM teachers, and the second author has contributed significantly to the development and refinement of the pedagogical tools which are the focus of this book. This intensive and extensive collaboration between the teachers and the authors has enabled ongoing development and review related to the Number Intervention F-4 Project. The authors wish to acknowledge the very significant contribution CEM teachers have made to the focus and content of this book. We also wish to acknowledge the following leaders and learning consultants from CEM for their significant roles in the Number Intervention F-4 Project: Judy Connell, Valerie Everist, David Huggins, Lucy Kett, Gerald Lewis, Simon Lindsay, Leanne Murray, Vicky Nally and Paul Sedunary.



Series Preface

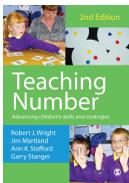
This book – *The Learning Framework in Number: Pedagogical Tools for Assessment and Instruction* is a significant and important addition to the current Mathematics Recovery Series. The six books in this series address the teaching of **early number**, whole number arithmetic and fractions in primary, elementary and secondary education. These books provide practical help to enable schools and teachers to give equal status to numeracy intervention and classroom instruction. The authors are internationally recognized as leaders in this field and draw on considerable practical experience of delivering professional learning programmes, training courses and materials.

The books are:



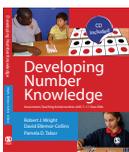
Early Numeracy: Assessment for Teaching and Intervention, 2nd edition, Robert J. Wright, Jim Martland and Ann K. Stafford, 2006.

Early Numeracy demonstrates how to assess students' mathematical knowledge, skills and strategies in addition, subtraction, multiplication and division.



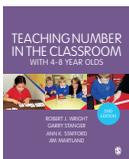
Teaching Number: Advancing Children's Skills and Strategies, 2nd edition, Robert J. Wright, Jim Martland, Ann K. Stafford and Garry Stanger, 2006.

Teaching Number sets out in detail nine principles which guide the teaching, together with 180 practical, exemplar teaching procedures to advance children to more sophisticated strategies for solving arithmetic problems.



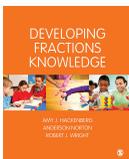
Developing Number Knowledge: Assessment, Teaching and Intervention with 7–11 Year-Olds, Robert J. Wright, David Ellemor-Collins and Pamela Tabor, 2012.

Developing Number Knowledge provides more advanced knowledge and resources for teachers working with older students.



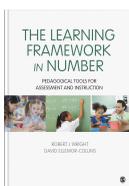
Teaching Number in the Classroom with 4–8 Year-Olds, 2nd edition, Robert J. Wright, Garry Stanger, Ann K. Stafford and Jim Martland, 2014.

Teaching Number in the Classroom shows how to extend the work of assessment and intervention with individual and small groups to working with whole classes.



Developing Fractions Knowledge, Amy J. Hackenberg, Anderson Norton and Robert J. Wright, 2016.

Developing Fractions Knowledge provides a detailed progressive approach to assessment and instruction related to students' learning of fractions.



The Learning Framework in Number: Pedagogical Tools for Assessment and Instruction, Robert J. Wright and David Ellemor-Collins, 2018.

This book presents a learning framework across the whole K to 5 range, and provides three sets of pedagogical tools for the framework – assessment schedules, models of learning progressions and teaching charts. These tools enable detailed assessment and profiling of children's whole number **arithmetic knowledge**, and the development of specific instructional programmes.

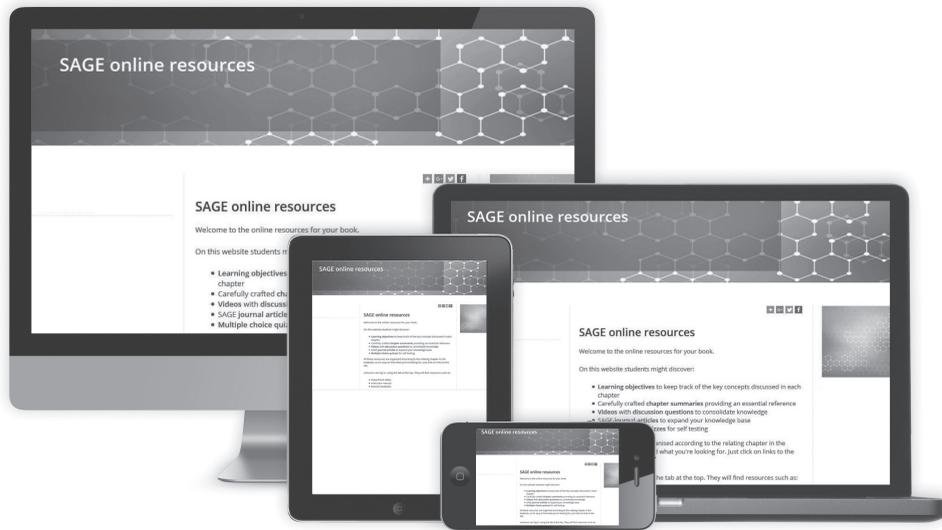
The series provides a comprehensive package on:

1. How to identify, analyse and report students' arithmetic knowledge, skills and strategies.
2. How to design, implement and evaluate a course of intervention.
3. How to include both assessment and teaching in the daily numeracy programme in differing class organizations and contexts.

The series draws on a substantial body of recent theoretical research supported by international practical application. Because all the assessment and teaching activities have been empirically tested the books are able to show the teacher the possible ranges of students' responses and patterns of their behaviour.

The books are a package for professional development and a comprehensive resource for experienced teachers concerned with intervention and instruction from Kindergarten to primary, elementary and secondary levels.

Online Resources



The Learning Framework in Number is supported by a range of downloadable resources available at: <https://study.sagepub.com/wrightLFIN>

These include:

- The downloadable **LFIN Assessment Kit** including printable resources necessary for carrying out the assessments detailed in the book
- The downloadable **LFIN Teaching Kit** including printable resources for use in classroom teaching



Introduction

This book presents a complete Learning Framework in Number (LFIN) across the primary grades K to 5. The LFIN provides a coherent overview of instruction in whole number arithmetic. It is organized into key **domains** of number from emergent numeracy through to strong additive and multiplicative arithmetic, which teachers can use to organize their assessment, planning and instruction.

Central to the LFIN are what we call *pedagogical tools*, that is, carefully designed materials that teachers can apply in their pedagogy. The LFIN incorporates a comprehensive suite of three main types of pedagogical tools: assessment schedules, models of learning progressions and teaching charts.

- An *assessment schedule* is a sequence of groups of **tasks** that teachers can use to assess students' levels of knowledge in a domain of number.
- A *learning progression* is presented as a table of levels that enables charting of students' current knowledge of a key aspect of number. We refer to each table of levels as a *model* of a learning progression.
- A *teaching chart* sets out interrelated progressions of instructional procedures to serve as a map for targeted, responsive instruction in a domain of number.
- Each of these tools is content-specific, and has developed from our ongoing work in teachers' professional learning.

Along with a broad consensus of mathematics educators, we do not regard learning number to be simply a task of memorizing facts or procedures. Rather, for each student, learning number demands a progression of increasing conceptual sophistication, along with the development of a rich network of number relationships. Supporting most students with this learning requires a responsive, targeted, **enquiry-based instruction** over several years of schooling. The LFIN and the suite of pedagogical tools are designed to enable such responsive instruction.

The purpose of the book

The purpose of this book is to provide the LFIN and pedagogical tools for teachers to use in their work with K to 5 students. Each tool is presented in a ready-to-use format, with sufficient instructions for teachers to use them, and with commentaries to highlight the details of students' number learning associated with each tool.

Using the book

The LFIN and pedagogical tools have been designed for intensive intervention. To this end, they are detailed. They can support subtle individual assessment and tailored, finely-graded individual instruction. Nevertheless, our experience is that teachers who learn to use these tools in intervention also bring the tools to their classroom teaching, and with great success. The core of the materials serves excellently for regular teaching. Teachers can skim the detail, and use the core tasks of the assessment schedules, the core levels of the models, and the core instructional progressions of the teaching charts. Meanwhile, the richer detail can still be of benefit at times in regular classroom teaching. So, all teachers of number and arithmetic can use the LFIN and tools in this book.

The tools are fully explained here, and the earlier books in the series furnish further compatible information on assessment and instruction. Nevertheless, to teach number well there is much to learn. We heartily recommend dedicated programmes of professional learning to make the most of these materials. For more on professional learning, see Purple Book Chapter 11 and Red Book Chapter 1. Note that these books are described in the latter part of this chapter.

We are convinced that these tools can contribute enormously to teachers' pedagogical knowledge and work, and so in turn, lead to much improved learning of arithmetic in our students and to a strong foundation for all their mathematics learning.

The background to the book

The tools and methods presented in this book are a culmination of an extensive and ongoing programme of research and development related to the teaching of number and arithmetic in the primary and elementary grades. In the case of the first author, this work spans at least 25 years, and includes several multi-year projects funded by the Australian Research Council (ARC) and a range of education departments and jurisdictions. One outcome of this work is Mathematics Recovery, a programme focusing on intensive intervention for low-attaining students in the early years of school. This programme was developed by the first author in New South Wales (Australia) and has been widely implemented internationally. As well, the programme has been extensively adapted to classroom teaching for average and able learners, and it provided a basis for earlier books in the series.

Since 2004, the authors have collaborated on a project focusing on intervention in the number learning of low-attaining 8- to 10-year-olds. The project is funded by the Australian Research Council (under Grant No. LP0348932) and Catholic Education Melbourne (CEM), and extends our earlier work on intensive intervention for younger students. Since 2012, we have also pursued a project on intervention in multiplication and division. The collaborative work of the authors of this book over the last 12 years, along with the first author's earlier work, provides a basis for the writing of this book.

With all of our research and development work, we have drawn extensively on a range of international research and development related to students' learning of arithmetic. Equally, our work depends on intensive, long-term collaborations with teachers, schools and school systems. We have drawn extensively on their insights into students' learning and purposeful instruction, as well as their trialling of the range of pedagogical tools that we have developed. This book in particular, is the outcome of these long-term collaborations with teachers.

The book should be of interest to all who are concerned with finding new ways to teach number and with advancing the levels of learning in schools. Teachers, advisers, numeracy consultants, mathematics supervisors, curriculum leaders and learning support personnel, as well as teacher educators and researchers whose work relates to this field, will find much of interest from both theoretical and practical perspectives.

The structure of the book

The book is structured using the basic order in which the pedagogical tools are learned and used by teachers. We begin with the organizing framework (LFIN). Then follows the approach to assessment and tools for assessment, and finally the approach to instruction and tools for instruction.

In Chapter 1 we provide an overview of the Learning Framework in Number (LFIN) and introduce the three types of pedagogical tools that are the focus of this book: assessment schedules, models of learning progressions and teaching charts.

In Chapter 2 we describe our approach to assessment. This includes an account of Video-recorded Interview-based Assessment (VIBA) and its particular usefulness in teacher professional learning; guidelines for conducting assessment interviews; guidelines for videotaping assessment interviews; and guidelines for coding and analysing assessment interviews.

In Chapter 3 we present the assessment schedules. We explain the format of the schedules, and how they are used to guide assessment interviews. We give an overview of the set of schedules. We then present the nine schedules, along with detailed commentaries for each schedule on students' responses to the assessment tasks.

In Chapter 4 we present the models of learning progressions. We explain the format of the models, and how they are used both to summarize assessment and to give direction to teaching. We give an overview of the set of models. We then present the nine models and, for each model, a detailed commentary on the significance of the levels and the sense of progression through the model.

In Chapter 5 we describe our enquiry-based approach to instruction. This includes guiding principles of instruction; key elements of instructional interaction in intervention; and characteristics of students' problem solving. Chapter 5 also includes a description of the broad trajectory of instruction in number across grades K to 5, and an overview of instruction in each domain.

Finally, in Chapter 6 we present the teaching charts. We explain the format of the charts, and how they are used to guide both moment-to-moment teaching and lesson planning. We give an overview of the set of teaching charts. We then present the 10 charts, along with commentaries for each chart on the main teaching procedures and progressions.

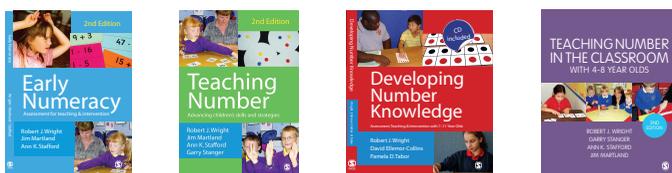
Useful reference sources are gathered at the end of the book, including:

- an extensive glossary of technical terms used in the book;
- an appendix which contains descriptions of key **instructional settings** used in the book;
- copies of two key charts from the book, the LFIN Chart and the Models Chart; and
- a comprehensive bibliography of relevant works.

The book's website has master documents for printing the materials in the LFIN Assessment Kit, and the LFIN Teaching Kit.

Using the earlier books in the series

This book is the sixth entry in the Math Recovery series which addresses assessment, instruction and intervention in early number and whole number arithmetic in the range from Kindergarten to 5th grade. Four earlier books are listed below. For ease of use, we refer to each book according to the cover colour: the Blue Book, the Green Book, the Purple Book and the Red Book.



Blue Book *Early Numeracy: Assessment for Teaching and Intervention*

Green Book *Teaching Number: Advancing Children's Skills and Strategies*

Red Book *Developing Number Knowledge: Assessment, Teaching and Intervention with 7–11 Year-Olds*

Purple Book *Teaching Number in the Classroom with 4–8 Year-Olds*

This book is an important addition to the series because for the first time it presents a learning framework across the whole K to 5 range, and provides a complete suite of pedagogical tools for that framework: assessment schedules, models of learning progressions and teaching charts. The new LFIN has been developed from earlier frameworks and organizers, to offer a coherent framework extending from preschoolers' emergent number knowledge through to the relatively sophisticated arithmetical knowledge typical of successful learners at 4th- and 5th-grade levels. The assessment schedules and models from our earlier books have been radically revised, reorganized, extended and furnished with new commentaries. The teaching charts are all new.

While an extensive amount of the material in this book is new to the series, the book relates closely to the four earlier books. Our strong recommendation to readers is that they use this book in conjunction with the earlier books. The earlier books provide detailed information related to many aspects of the pedagogical tools in this book. The earlier books also offer more extensive descriptions of our general approach to assessment and instruction. Thus, in the earlier books, the reader will find additional information that can be applied in coming to understand the content and significance of all of the pedagogical tools. To this end, a table in Chapter 1 lays out the links from this book to the chapters of the earlier books. Cross-references to the earlier books are provided throughout this book.

Developing fractions knowledge

Finally, please note that, as well as this new book and the four books listed above, a sixth book in the Mathematics Recovery Series, entitled *Developing Fractions Knowledge*, is also available.

